1 Motivation

Consider a classical ODE

$$\dot{Y}_t = f(t, Y_t) \quad Y_0 = y \in \mathbb{R}^n$$

Then we know that $\exists!$ solution if f is Lipschitz in Y_t (in L^1_t way). An important subclass are *controlled* ODEs which are of the form

$$\dot{Y}_t = f(Y_t)\dot{X}_t \quad Y_0 = y$$

for X, f, and y given. For nice enough f and X,

$$Y_t = y + \int_0^t f(Y_s) dX_s$$

proceed by Picard iteration, find a fixed point. We can think of X as an input. We are interested in the map $X \mapsto Y$. Formally, $\Delta Y = f(Y)\Delta X$ or $dY_t = f(Y_t)dX_t$.

- What if X is rough? $X \in C^\alpha, \quad 0 < \alpha < 1$. For smooth f, we should expect Y inherits same regularity as $X \to Y \in C^\alpha$
- Strategy: $f^\epsilon \to f$, $g^\epsilon \to g$ where $f^\epsilon, g^\epsilon \in C^\infty$.

$$\int_0^t f_s^\epsilon \dot{g}_s^\epsilon \, ds \to \int_0^t f_s dg_s$$

1. If $f^\epsilon \to f$ uniformly ($f \in C$) and $\dot{g}^\epsilon \to \dot{g}$ uniformly ($g \in C^1$), then this integral makes sense.
2. If $f^\epsilon \to f$ uniformly ($f \in C$) and $\dot{g}^\epsilon \to \dot{g}$ in L^1 ($g \in W^{1,1}$)

$$\left| \int f_s^\epsilon \dot{g}_s^\epsilon \, ds - \int f_s \dot{g}_s \, ds \right| \le \left| \int_0^t f_s^\epsilon (\dot{g}_s^\epsilon - \dot{g}_s) \, ds + \int_0^t (f_s^\epsilon - f_s)\dot{g}_s \, ds \right| \tag{1}$$

$$\le C||\dot{g}^\epsilon - \dot{g}||_1 + ||f^\epsilon - f||_\infty ||\dot{g}||_1 \to 0 \tag{2}$$

Further, integrating by parts, we see

$$\int_0^t f_s \dot{g}_s \, ds = f_t g_t - f_0 g_0 - \int_0^t g_s \dot{f}_s \, ds$$

so $f \in W^{1,1}$ and $g \in C$ also works. Stronger: $f \in C^1, \; g \in C$.

- Can we interpolate? $f \in C^\alpha, g \in C^{1-\alpha}$? NOPE.

1.1 Young's integration theory

We can approximate the integral

$$\int_0^1 f_s dg_s = \lim_{|\Delta| \to 0} \sum_{i=1}^{N-1} f_{t_i}(g_{t_{i+1}} - g_{t_i})$$

where $\Delta = \{0 = t_0 < t_1 < \cdots < t_N = 1\}$ and $|\Delta| = \max\{|t_{i+1} - t_i|\}$. For a small interval $[s, t]$,

$$\int_s^t f_r dg_r = f_s(g_t - g_s) + R_{s,t} = f_s \delta g_{s,t} + R_{s,t}$$

where the remainder $R_{s,t}$ is of higher order. For $f \in C^\alpha$ and $g \in C^\beta$, $\alpha, \beta > 0$ what can we say?

$$R_{s,t} = \int_s^t (f_r - f_s) dg_r = \delta f_{t,s} \delta g_{t,s} + R'_{t,s}$$

where hopefully $R'_{t,s}$ is of hopefully even higher order. Then since $f \in C^\alpha$ and $g \in C^\beta$,

$$R_{s,t} \le [f]_\alpha [g]_\beta |t - s|^{\alpha+\beta} + R'_{s,t}$$

Then, considering the whole integral as the sum of integrals over the small intervals $[t_i, t_{i+1}]$,

$$\left| \int_0^1 f_r dg_r - \sum_{i=0}^{N-1} f_{t_i}(g_{t_{i+1}} - g_{t_i}) \right| \le [f]_\alpha [g]_\beta \sum_{i=0}^{N-1} |t_{i+1} - t_i|^{\alpha+\beta} \tag{3}$$

$$= [f]_\alpha [g]_\beta \sum_{i=0}^{N-1} |t_{i+1} - t_i|^{\alpha+\beta-1} |t_{i+1} - t_i| \tag{4}$$

$$\le [f]_\alpha [g]_\beta |\Delta|^{\alpha+\beta-1} \to 0 \quad (|\Delta| \to 0) \tag{5}$$

for $\alpha + \beta > 1$ so that $\alpha + \beta - 1 > 0$.

Theorem 1.1. (Young's integration) If $\alpha + \beta > 1$, then $\int_0^1 f_s dg_s = \lim_{|\Delta| \to 0} \sum_{i=0}^{N-1} f_{t_i}(g_{t_{i+1}} - g_{t_i})$ exists for $f \in C^\alpha$ and $g \in C^\beta$ and $(f, g) \mapsto \int_0^t f_s dg_s$ is bilinear and continuous* on $C^\alpha \times C^\beta$.

Note: C^α, C^β are not separable spaces. We mean continuity in $f_n \to f$ uniformly with bounded seminorm $\sup_{n\in\mathbb{N}}[f]_\alpha < \infty$

We can think of this integral $\int_0^1 f_r dg_r = \langle f\dot{g}, 1_{[s,t]}\rangle$ as testing $f\dot{g}$ against $1_{[s,t]}$ (which we can approximate by test functions).

$$Y_t = f(Y_t)dX_t, \quad Y_0 = y \to Y_t = y + \int_0^t f(Y_s)dX_s \quad \text{(YDE)} \tag{6}$$

For f smooth and $X \in C^\alpha$, we can hope (at best) $Y \in C^\alpha$. Then, $f(Y) \in C^\alpha$. Thus, for $\alpha+\alpha > 1 \to \alpha > 1/2$, we can use Young's integration.

Theorem 1.2. For $\alpha > 1/2$, f smooth, $X \in C^\alpha$, then $\exists!$ solution Y to YDE and $X \mapsto Y$ is locally lipschitz continuous.

1.2 Brownian motion as a motivating example

$\mathbb{E}[(B_t - B_s)^2] = t - s \to$ "$B_t - B_s \sim |t-s|^{1/2}$" that is the BM has regularity less than 1/2.

Theorem 1.3. For $0 < \alpha < 1$,

1. If $\alpha < 1/2$, $B \in C^\alpha$ a.s.
2. If $\alpha \geq 1/2$, $B \notin C^\alpha$ a.s. (not even locally) $\to$ can't do Young's integration
3. B is nowhere differentiable and infinite variation a.s. $\to$ can't use BV theory

So, we need a new theory of integration to deal with Brownian motion paths.

Consider again our ODE, but we allow X to be brownian motion.

$$\begin{cases} dY_t = f(Y_t)dX_t \\ Y_0 = y, X_0 = 0 \end{cases}$$

X and Y could be in $\mathbb{R}^d$ or infinite dimensional. Ex:

$$\begin{cases} dY_t^1 = dX_t^1 \\ dY_t^2 = Y_t^1 dX_t^2 \\ y = 0 \end{cases} \to \begin{cases} Y_t^1 = X_t^1 \\ Y_t^2 = \int_0^t X_s^1 dX_s^2 \end{cases}$$

If X^i are BM, then Young integral or Riemann-Stieltjes integration do not work since X^i is not regular enough.

We can think of the Brownian motion in two ways: **(1)** as a process B_t that a sequence of real-valued random variables, i.e. at every time t, B_t is a real-valued random variable, or **(2)** as a random variable on the space of paths, i.e. for each $B(\omega)$ is an entire path. From this second interpretation, we have the definition of a Wiener measure, that is, the probability distribution on the space of continuous functions g with $g(0) = 0$ induced by a Brownian motion.

Let μ be a Wiener measure on $C[0,1]$. That is, $\mu(A) = \mathbb{P}[B(\omega) \in A]$ for $A \subset [0,1]$.

Theorem 1.4. There exists no separable Banach space $\mathcal{B} \subset C[0,1]$ such that

1. μ is supported on $\mathcal{B}$
2. $(f,g) \mapsto \int_0^1 f(t)g'(t)\,dt$ on $C \times C^1$ extends continuous to $\mathcal{B} \times \mathcal{B}$

We can think of $\mathcal{B}$ as a regularity class constraint. This theorem says that there are no $\mathcal{B}$ on which we can define $\int_0^1 f dg$ that also contains almost all the paths of a BM.

Recall the Stieltjes integral

$$I := \int_0^1 X_s^1 dX_s^2 = \lim_{|\Delta|\to 0} \sum_{[s,t]\in\Delta} X_s^1 \delta X_{s,t}^2$$

Take $\{\Delta_n\}$ to be the sequence of dyadic partitions (nested and equally spaced)

$$I_n := \sum_{k=0}^{2^n-1} X_{k/2^n}^1 \delta X_{k/2^n,(k+1)/2^n}^2$$

Let $A_{s,t} = X_s^1 \delta X_{s,t}^2$

$$\begin{aligned} I_n - I_{n+1} &= \sum_{k=0}^{2^n-1} \left[A_{k/2^n,(k+1)/2^n} - A_{k/2^n,(2k+1)/2^{n+1}} - A_{(2k+1)/2^{n+1},(k+1)/2^n}\right] \\ &= \sum_{k=0}^{2^n-1} \delta X_{k/2^n,(2k+1)/2^{n+1}}^1 \delta X_{(2k+1)/2^{n+1},(k+1)/2^n}^2 \end{aligned}$$

Since $\mathbb{E}[(\delta X_{s,t}^i)^2] = t - s$

$$\approx \sum_{k=0}^{2^n-1} |t_{n+1} - t_n| = O(1)$$

If we use Young's inequality to get $(\delta X^i)^2$ we find that the I_n are not Cauchy. We should show $\{I_n\}$ Cauchy in a different norm.

$$\mathbb{E}[(I_n - I_{n+1})^2] = \mathbb{E}\left[\sum_{j,k}^{2^n-1} \delta X^1_{j/2^n,j+1/2^n}\delta X^1_{k/2^n,2k+1/2^{n+1}}\delta X^2_{2j+1/2^{n+1}}\delta X^2_{2k+1/2^{n+1},k+1/2^n}\right] \tag{7}$$

$$= \sum_{k=0}^{2^n-1}\left(\mathbb{E}\left[\delta X^1_{k/2^n,2k+1/2^{n+1}}\right]\right)^2\left(\mathbb{E}\left[\delta X^1_{2k+1/2^{n+1},k+1/2^n}\right]\right)^2$$

by independence of increments, only $j = k$ terms remain.

$$= \sum_{k=0}^{2^{-n-1}1} 2^{-n-1}\cdot 2^{-n-1} = \frac{1}{4}\sum_{k=0}^{2^n-1} 2^{-n} \approx 2^{-n}$$

So,

$$||I_n - I_{n+1}||_{L^2(\Omega)} \le C2^{-n/2}$$

Now, suppose $X^1 = X^2 = X$, then we expect $\int_0^t X_s dX_s = \frac{1}{2}X_t^2$.

"Proof." IBP

$$\int_0^t X_s dX_s = X_t^2 - \int_0^t X_s dX_s \Rightarrow \int_0^t X_s dX_s = \frac{1}{2}X_t^2 \quad \text{“}\square\text{”}$$

Repeat the partition argument:

$$I_\Delta = \sum_{[s,t]\in\Delta} X_s\delta X_{s,t}, \quad \mathbb{E}[I_\Delta] = \sum_{[s,t]\in\Delta}\mathbb{E}[X_s]\mathbb{E}[\delta X_{s,t}] = 0$$

since X_s and $\delta X_{s,t}$ are independent. However, $\mathbb{E}\left[\frac{1}{2}X_t^2\right] = \frac{1}{2}t^2$. In reality, in $L^2(\Omega)$ limit,

$$\lim_{|\Delta|\to 0} I_\Delta = \frac{X_t^2}{2} - \frac{t}{2}$$

<u>Conclusion</u>: Stochastic integral does not satisfy IBP, Chain Rule, Product Rule, etc. In classical calculus, quadratic size increments go to 0, but for BM quadratic size increments go to dt

What, instead of the left endpoint, we take the midpoint?

$$\tilde{I}_\Delta = \sum_{[s,t]\in\Delta} X_{s+t/2}\delta X_{s,t}$$

$\tilde{I}_\Delta$ also has limit $\frac{X_t^2}{2}$. This is the Stratonovich integral $\int_0^t X_s \circ dX_s$.

Takeaways

1. There might not be an analytically unique choice of solution
2. The iterated integrals $\int_0^t X_s^i dX_s^i$ are important

2 Rough Paths

Moving forward, we are concerned with the following type of problem which we name a *Rough Differential Equation* (RDE)

$$(RDE) \quad \begin{cases} dY_t = f(Y_t)dX_t \\ Y_0 = y \end{cases} \tag{8}$$

where $X \in C^\alpha$ for $1/3 < \alpha \le 1/2$.

We can express Y_t as integral, $Y_t = y + \int_0^t f(Y_s)dX_s$ (we always assume $f \in C^\infty$). Taking a small interval, we can approximate this integral

$$\int_s^t f(Y_r)dX_r = f(Y_s)\delta X_{s,t} + \int_s^t [f(Y_r) - f(Y_s)]dX_r$$

However, this error term is not better than linear for $\alpha \le 1/2$ since $f(Y_t) \in C^\alpha$ and $X \in C^\alpha$.

Taylor expand f:

$$\int_s^t f(Y_r)dX_r = f(Y_s)\delta X_{s,t} + \int_s^t\int_s^r Df(Y_q)dY_q dX_r \tag{9}$$

$$= f(Y_s)\delta X_{s,t} + \int_s^t\int_s^r Df(Y_q)f(Y_q)dX_q dX_r \tag{10}$$

$$= f(Y_s)\delta X_{s,t} + Df(Y_s)f(Y_s)\int_s^t\int_s^r dX_q dX_r + \int_s^t\int_s^r [(fDf)(Y_q) - (fDf)(Y_s)]dX_q dX_r \tag{11}$$

$$= f(Y_s)\delta X_{s,t} + Df(Y_s)f(Y_s)\int_s^t \delta X_{s,r}dX_r + \int_s^t\int_s^r [(fDf)(Y_q) - (fDf)(Y_s)]dX_q dX_r \tag{12}$$

where the last error term is $\lesssim |t-s|^{3\alpha}$. This gives a new boundary $\alpha > 1/3$, but lets us address $\mathbf{1/3} < \alpha \leq \mathbf{1/2}$. That is, for $\alpha > 1/3$, up to $o(|t-s|)$,

$$\int_s^t f(Y_r)dX_r = f(Y_s)\delta X_s + f(Y_s)Df(Y_s)\mathbb{X}_{s,t}$$

where $\mathbb{X}_{s,t} = \int_s^t \delta X_{s,r} \otimes dX_r$. **Remember these are tensors**

We can define a lift $\mathscr{L}$ from $X \in C^\alpha$ to $(X, \mathbb{X}) \in \mathscr{C}^\alpha$ the rough path space that is

- nonunique
- often requires probability
- universal, i.e. independent of function f or initial condition y

From this lift $\mathscr{L}$ we can define a map $\mathcal{S} : \mathscr{C}^\alpha \to C^\alpha$ given by $(X, \mathbb{X}) \mapsto Y$ that is

- unique
- continuous with respect to the right metric (we cannot expect it to be continuous in C^α)

We can repeat this procedure to lower the threshold of viable α regularity, but it becomes more complicated with each new layer as we need to understand what it means to integral $\mathbb{X}$ against X in the next step for example. We will generally take $1/3 < \alpha \leq 1/2$.

2.1 White Noise

Definition 2.1 (White Noise)**.** ξ is random distribution (in the analysis sense) that is a centered Gaussian such that

$$\mathbb{E}[\langle \xi, \phi\rangle\langle \xi, \psi\rangle] = \int \phi(x)\psi(x).$$

2.1.1 Negative Hölder Continuity

We can't use the same method to measure regularity as for positive Hölder spaces. $|f(x) - f(y)| \lesssim |x-y|^{-\alpha}$ doesn't makes sense since as $x-y$ gets small, $|x-y|^{-\alpha}$ blows up. So, we instead measure how the distribution behave against scaled test functions. Let $\varphi_\lambda = \lambda^{-d}\varphi(x\lambda^{-1})$. If f is a function,

$$\langle f, \varphi_\lambda\rangle = \frac{1}{\lambda^d}\int f(x)\varphi(x/\lambda)\ dx = \int f(\lambda x)\varphi(x)\ dx.$$

For nice functions, this smooths them–for negative regularity, blow up. Thus, if $f \in C^{-\alpha},\ \alpha > 0$,

$$|\langle f, \varphi_\lambda\rangle| \lesssim \lambda^{-\alpha} \quad ||f||_{-\alpha} = \sup_{\varphi\in B}\sup_{0<\lambda<1} \lambda^\alpha|\langle f, \varphi_\lambda\rangle|.$$

So, for white noise,

$$\mathbb{E}[|\langle \xi, \varphi_\lambda\rangle|^2] = \int \varphi_\lambda(x)^2\ dx = \frac{1}{\lambda^{2d}}\int \varphi(x/\lambda)^2\ dx = \frac{1}{\lambda^d}||\varphi||_2^2.$$

Thus, $\langle \xi, \varphi_\lambda\rangle \approx \lambda^{-d/2}$ so regularity of ξ is a bit worse than $-d/2$, $\xi \in C^{-d/2-}$.

2.1.2 Stochastic PDE examples

1. Stochastic Heat Equation

$$\begin{cases} \partial_t u = \Delta u + \xi \\ u|_{t=0} = u_0 \end{cases} \quad \Rightarrow u(t,x) = \Phi * u_0 + \int_0^t \Phi_{t-s} * \xi(s,\cdot)\ ds$$

If we take $d = 1$,

$$\partial_t u = \partial_x^2 u + \xi \Rightarrow Pu = \xi, \quad P = \partial_t - \Delta$$

We think of this instead as a parabolic operator, to get parabolic regularity scale

$$\varphi_\lambda(t,x) = \lambda^{-3}\varphi(\lambda^{-2}t, \lambda^{-1}x)$$

so

$$|\langle \xi, \varphi_\lambda\rangle| \lesssim \lambda^{-3/2-} \Rightarrow \xi \in C_p^{-3/2-}$$

meaning $u \in C_p^{1-3/2-} = C_p^{1/2-} = C_t^{1/4-}C_x^{1/2-}$. So, $\int_0^t \Phi_{t-s} * \xi(s,\cdot)\ ds$ is a function, but we still need to think about it in the weak sense.

In general, $u \in C_t^{\left(\frac{1}{2}-\frac{d}{4}\right)-}C_x^{\left(1-\frac{d}{2}\right)-}$

2. "Burgers-like equation"

$$\partial_t u^i = \partial_x^2 u^i + g(u)\partial_x u^i + \xi^i, \quad i = 1, 2, \dots, n \quad (g \text{ smooth}) \tag{13}$$

$$\Rightarrow -\int u d\varphi = \int u\partial_x^2\varphi + \int g(u)\partial_x u\varphi + \langle \xi, \varphi\rangle \tag{14}$$

Why can't we define $\int f dg = \int f g'\ dt$ where g' is the distributional derivative?

We *usually* need $f \in C_c^\infty$ to do this. However, we can do it if $f \in C^\alpha$ and $g' \in C^{\beta-1}$ so long as $\alpha + \beta > 1$

Note: $C^{-\alpha} = B^{-\alpha}_{\infty,\infty}$ is a Besov space.

ξ^i is white noise on $\mathbb{R} \times (0, \infty) \in C_p^{-3/2-} = C_t^{-3/4-} C_x^{-3/2-}$ so we expect $u^i \in C_p^{1/2-} = C_t^{1/4-} C_x^{1/2-}$. Note that $C_x^{1/2-}$ is the same regularity as a BM.

- What about $g(u)\partial_x u^i$? In general, we can't think of $g(u)\partial_x u = \partial_x[G(u)]$

$g : \mathbb{R}^n \to \mathbb{R}^n$ smooth, then $g(u) \in C_x^{1/2-}$ and $\partial_x u^i \in C_x^{-1/2-}$ so we're just barely out of the regime where we can do $\int fg'$ via distributions. (!!)

- Idea: cancel out singular part of equation: the white noise ξ^i.

$$\begin{cases} \partial_t \psi^i = \partial_{xx}\psi^i + \xi^i \\ \psi^i\Big|_{t=0} = 0 \end{cases}$$

We want to decompose u into [rough] + [less rough]. Let $u^i = v^i + \psi^i$ where

$$\partial_t v_i = \partial_{xx} v^i + g(v+\psi)\partial_x v^i + g(v+\psi)\partial_x \psi^i$$

We think v should be more regular.

Recall: $\int_0^t f_r dg_r \in C_t^\beta$ where $g \in C_t^\beta$ Then, $g(v+\psi)\partial_x\psi^i \in C_x^{-1/2-}$ so we expect $v \in C_x^{1,1/2-}$. Thus, $g(v+\psi)\partial_x v^i$ makes sense as a function.

- Let $\varphi \in C_c^\infty(\mathbb{R})$ and consider $\int_{\mathbb{R}} g(v+\psi)\partial_x\psi\varphi\ dx$. We can't do IBP since we will get $\partial_x\psi$.

$$\int_y^{y+h} g(v+\psi)\partial_x\psi\varphi \approx g(v(y)+\psi(y))\delta\psi_{y,y+h}\varphi(y) \tag{15}$$

$$= G(y,\psi(y))\delta\psi_{y,y+h} + \int_y^{y+h} [G(x,\psi(x)) - G(y,\psi(y))]\ dx\psi(x) \tag{16}$$

$$= G\delta\psi_{y,y+h} + \int_y^{y+h} \left[\int_y^x [(\partial_x G)(z,\psi(z))\ dz + \int_y^x \partial_\psi G)(z,\psi(z))\ dz\psi(z)\right] dx\psi(x)] \tag{17}$$

$$= G\delta\psi + \underbrace{\int_y^{y+h}\int_y^z \underbrace{\partial_x G(z,\psi)}_{\text{bounded}} dzdx\psi(x)}_{\approx O(h^{3/2}) \text{ acceptable error}} + \int_y^{y+h}\int_y^x \partial_\psi G(z,\psi)\ \underbrace{dz\psi(z)dx\psi(x)}_{\approx O(h^{-1})} \tag{18}$$

$$= (\partial_\psi G)(y,\psi(y))\int_y^{y+h}\int_y^x dz\psi(z)\ dx\psi(x) + o(h) \tag{19}$$

where $G(x,\psi(x)) = g(v(x)+\psi(x))\varphi(x) \in C^1$ So, we need to make sense of

$$\Psi_{y,y+h} = \int_y^{y+h}\int_y^x dz\psi(z)\ dx\psi(x) = \int_y^{y+h} (\psi^i(t,x) - \psi^i(t,y))\ dx\psi^j(t,x).$$

2.2 Lift to the space of Rough Paths

Let $X : [0,T] \to V = \mathbb{R}^m$ (or any separable Banach space in most cases). We want to define what

$$\mathbb{X}_{s,t} = \int_s^t \delta X_{s,r} \otimes dX_r$$

means. We think of $\mathbb{X}_{s,t} = I(\delta(X_{s,\cdot}))$ where $f \mapsto I_{s,t}(f) = \int_s^t f_r dX_r$ We require that $I_{s,t}$ satisfy the following:

1. $I_{s,t}$ is linear
2. $I_{s,t}(1) = \delta X_{s,t}$
3. If $s < t < u$, $I_{su} = I_{st} + I_{tu}$. This implies

$$\begin{aligned} \mathbb{X}_{su} = I_{su}(\delta X_{s,\cdot}) &= I_{s,t}(\delta(X_{s,\cdot}) + I_{tu}(\delta X_{s,\cdot}) \\ &= \mathbb{X}_{s,t} + I_{tu}(\delta X_{s,t} + \delta X_{t,\cdot}) \\ &= \mathbb{X}_{s,t} + \mathbb{X}_{t,u} + \delta X_{s,t} \otimes \delta X_{t,u} \end{aligned} \tag{20}$$

Define $\mathbb{X}$ such that Chen's relation (20) hold. Define

$$\Delta_T^n = \{(s_1,\dots,s_n) \in [0,T]^n : s_1 \le \dots \le s_n\} \Rightarrow \Delta_T^1 = [0,T],\ \Delta_T^2 \subset [0,T]^2 = \{s \le t\},\ \Delta_T^3 = \{s \le t \le u\},\dots$$

So we can think of δ as $C(\Delta_T^1) \xrightarrow{\delta} C(\Delta_T^2) \xrightarrow{\delta} C(\Delta_T^3)$ This sequence is *short* and *exact*.

$$\delta^2 = 0, \quad \delta A_{s,t} = 0 \Rightarrow A_{s,t} = \delta F_{s,t} \text{ for some } F$$

δ eats functions of (s,t) and spits out functions of (s,t,u). In general, takes n variable functions and returns $n+1$ variable functions.

$X \in C^\alpha, \frac{1}{3} < \alpha \le \frac{1}{2}$ "Generic" Holder paths should be "self-similar". $X_{\lambda t} - X_{\lambda s} \sim \lambda^\alpha \delta X_{s,t}$

$$\mathbb{X}_{\lambda s,\lambda t} = \int_{\lambda s}^{\lambda t} (X_r - X_{\lambda s}) \otimes dX_r \quad (\text{ let } r = \lambda q) \tag{21}$$

$$= \lambda \int_s^t (X_{\lambda q} - X_{\lambda s}) \otimes dX_{\lambda q} \tag{22}$$

$$\sim \lambda \cdot \lambda^\alpha \int_s^t (X_q - X_s) dX_q \quad \text{since } X \in C^\alpha \tag{23}$$

$$\sim \lambda \lambda^\alpha \lambda^{\alpha-1} \int_s^t (X_q - X_s) dX_q \quad \text{since } dX \in C^{\alpha-1} \tag{24}$$

$$= \lambda^{2\alpha} \mathbb{X}_{s,t} \tag{25}$$

So, we want $\mathbb{X}$ to be twice a regular as X.

Definition 2.2. Let $\frac{1}{3} < \alpha \le \frac{1}{2}$. Then $(X,\mathbb{X})$ is an *α-Holder rough path* if $X : \Delta_T^1 \to V$ and $\mathbb{X} : \Delta_T^2 \to V \otimes V$ satisfy

1. $\mathbb{X}_{su} - \mathbb{X}_{st} - \mathbb{X}_{tu} = \delta X_{s,t} \otimes X_{t,u}$, Chen's relation
2. $[X]_\alpha = \sup_{(s,t)\in\Delta_T^2} \frac{|X_t - X_s|}{|t-s|^\alpha} < \infty$ and $||\mathbb{X}||_{\mathbb{C}^{2\alpha}} = \sup_{(s,t)\in\Delta_T^2} \frac{\mathbb{X}_{s,t}}{|t-s|^{2\alpha}} < \infty$

This says that $\mathbb{X}_{s,t}$ is small when $|t-s|$ is small.

Note: $\mathbb{C}^{2\alpha}$ is used instead of $C^{2\alpha}$ because $\mathbb{X}$ is a 2 variable function and $C^{2\alpha}$ sounds like $\mathbb{X}$ is $C^{2\alpha}$ in each variable.

Proposition 2.1. $\mathbb{X} \in \mathbb{C}^{2\alpha}(\Delta_T^2)$

Proof.

$$|\mathbb{X}_{s,t_1} - \mathbb{X}_{s,t_2}| = |\mathbb{X}_{t,t_2} + \delta X_{s,t_1} \delta X_{s,t_2}| \lesssim |t_1 - t_2|^{2\alpha} + C|t_1 - t_2|^\alpha$$

Similarly for s_1, s_2 ☐

- *Example 1*: If $X \in C^1(\Delta_T^1; V)$, and we define $\mathbb{X}_{s,t} = \int_s^t \delta X_{s,r} \dot{X})_r dr$, then $(X,\mathbb{X}) \in C^\alpha([0,T];V)$ is a rough path.
- *Example 2*: Let $X \equiv 0$. Let $A \in V \otimes V$ $(A \in R^m \otimes \mathbb{R}^m)$ then $(0,(t-s)A) \in C^{1/2}([0,T],V)$. That is $\mathbb{X}_{s,t} = (t-s)A$ We'll see that $\mathbb{X}$ corresponds to *areas* traced out by curves Ex 3: Brownian sample paths $B \in C^\alpha\ \forall\ \alpha < 1/2$ a.s. Define $\mathbb{B}_{s,t}^I = \int_s^t \delta B_{r,s} \otimes dB_r := \lim_{|P|\to 0} \sum_i \delta B_{t_i,s} \delta B_{t_i,t_{i+1}}$ this is the Ito integral. Then $(B,\mathbb{B}^I) \in C^\alpha\ \forall\ \alpha < 1/2$

Proposition 2.2. There exists a (nonunique) map $\epsilon : C^\alpha(\Delta_T^1) \to \mathbb{C}^{2\alpha}(\Delta_T^2), \quad X \mapsto \mathbb{X}$ such that $(X,\epsilon X) \in C^\alpha$ This is called the *Lyons-Victoir Extension*

2.3 Topology

We define the following metrix on the space of rough paths, $\mathscr{C}^\alpha$,

$$\rho_\alpha(\mathbb{X},\mathbb{Y}) = [X - Y]_\alpha + ||\mathbb{X} - \mathbb{Y}||_{\mathbb{C}^{2\alpha}}$$

We want $\rho_\alpha(\mathbb{X},\mathbb{Y}) = 0 \Rightarrow \mathbb{X} = \mathbb{Y}$ so enforce $X = 0$. Then $(\mathscr{C}^\alpha, \rho_\alpha)$ is a complete metric space. We define the "norm"

$$|||(\mathbb{X},X)|||_{\mathscr{C}^\alpha} = [X]_{C^\alpha} + \sqrt{||\mathbb{X}||_{\mathbb{C}^{2\alpha}}}$$

Note: the square root is for "homogeneity".

Note : $\mathscr{C}^\alpha$ is not a linear space and this "norm" does not satisfy the triangle inequality.

Define $\delta_\lambda : \mathscr{C}^\alpha \to \mathscr{C}^\alpha, \lambda > 0$ by

$$(X,\mathbb{X}) \mapsto (\lambda X, \lambda^2 \mathbb{X})$$

So, $|||\delta_\lambda(X,\mathbb{X})|||_{\mathscr{C}^\alpha} = \lambda|||(X,\mathbb{X})|||_{\mathscr{C}^\alpha}$

2.3.1 Smooth Approximations

Note : C^1 is not dense in C^α, C^α is not separable.

Define the C^α norm by $||X||_\alpha = ||X||_\infty + [X]_\alpha$ or $X_0 + [X]_\alpha$. Then $C^\alpha \supset C_0^\alpha := \overline{C^1}^{C^\alpha}$ $X \in C_0^\alpha$ iff $\frac{|X_t - X_s|}{|t-s|^\alpha} \to 0$ as $|t-s| \to 0$.

Example: $X_t = t^\alpha \in C^\alpha$ but $t^\alpha \notin C_0^\alpha$ $\forall\ \beta > \alpha$: $C^\beta \subset C_0^\alpha \subset C^\alpha$ (this inclusion is continuous)

If $X_n \to X$ uniformly and $[X_n]_\beta \le C$, then $||X_n - X||_\alpha \to 0\ \forall\ \alpha < \beta$

Example: BM $\in C^\alpha$ a.s. $\forall\ \alpha < \frac{1}{2}$. Thus, BM $\in C_0^\alpha$ a.s. $\forall\ \alpha < 1/2$ In fact, $B \in \mathcal{B}^{1/2}_{p,\infty}$ $\forall\ p < \infty$. (sharp: p cannot be increased and ∞ cannot be lowered)

Question: Given $(X, \mathbb{X}) \in \mathscr{C}^\alpha$, does there exists $X^n \in C^1([0,T], V)$ such that if $\mathbb{X}^n_{s,t} := \int_s^t \delta X^n_{s,r} \otimes dX^n_r$, then $\rho_\alpha((X^n, \mathbb{X}^n), (X, \mathbb{X})) \to 0$? Or if not, do we at least have

$$||X^n - X||_\infty + ||\mathbb{X}^n - \mathbb{X}||_\infty \to 0$$

and $\sup_n |||(X, \mathbb{X})|||_{\mathscr{C}^\alpha} < \infty$?

Answer: In general, no.

Let X smooth.

$$\mathbb{X}^{i,j}_{s,t} = \int_s^t (X^i_r - X^i_s) dX^j_r = (X^i_t - X^i_s)(X^j_t - X^j_s) - \underbrace{\int_s^t (X^j_r - X^j_s) dX^i_r}_{\mathbb{X}^{j,i}_{s,t}}$$

Thus, $\mathrm{Sym}(\mathbb{X}_{s,t}) = \frac{1}{2}\delta X_{s,t} \otimes \delta X_{s,t}$

Recall, $A = \frac{A + A^T}{2} + \frac{A - A^T}{2}$ decomposition into symmetric and antisymmetric components.

Antisymmetric component has something to do with the area, think $A = \frac{1}{2}\int xdy - ydx$

Definition 2.3. $\frac{1}{3} < \alpha \le \frac{1}{2}$, $(X, \mathbb{X}) \in \mathscr{C}^\alpha$ is a geometric $\alpha-$Holder Rough Path if $\mathrm{Sym}(\mathbb{X}_{s,t}) = \frac{1}{2}\delta X_{s,t} \otimes \delta X_{s,t}$ holds. We say $(X, \mathbb{X}) \in \mathscr{C}_g^\alpha$.

That is $\mathrm{Sym}(\mathbb{X}_{s,t})$ is determined by X. However, antisymmetric part has freedom.

Example 2.1. (Geometric rough path as limits of smooth approximations)

$X^n_t = \alpha n^{-1/2}(\cos(2\pi nt), \sin(2\pi nt))$, $\mathbb{X}^n_{s,t} = \int_s^t (X^n_r - X^n_s) \otimes X^n_r\ dr$. Then

$$(X^n, \mathbb{X}^n) \overset{\text{uniformly}}{\to} \left(0, \frac{\alpha^2}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}(t-s)\right) \in \mathscr{C}_g^{1/2}$$

Let $\mathbf{X} = (X, \mathbb{X})$. Since our goal is $dY = f(Y)dX$, consider $\dot{Y}^n = f(Y^n)\dot{X}^n$. Then if $\mathbf{X}^n \to \mathbf{X}$. Then $Y^n \to Y$ where $dY = f(Y)d\mathbf{X}$.

Example 2.2. Let $f : \mathbb{R} \to \mathbb{R}^2$, $x \mapsto (f_1(x), f_2(x))$.

$$Y_t - Y_s = f_1(Y_s) \cdot \delta X^1_{s,t} + f_2(Y_s) \cdot \delta X^2_{s,t} + tr\left[\begin{pmatrix} f_1 f_1' & f_2 f_1' \\ f_1 f_2' & f_2 f_2' \end{pmatrix}(Y_s)\mathbb{X}_{s,t}\right] + o(t-s)$$

Increments of the base path are zero (?)

$$Y_t - Y_s = \frac{\alpha^2}{2}\{f_1, f_2\}(Y_s)$$

where $\{f_1, f_2\} = f_1 f_2' - f_2 f_1'$ is the Poisson bracket. We can choose paths X^n that go to zero but their effects do not go to zero. If flows f_1, f_2 do not commute, then effects are nontrivial

2.4 Brownian motion as a rough path

Loosely, $\mathbb{E}[(B_t - B_s)^2] = m(t-s) \to \mathbb{E}\left[\left(\frac{B_t - B_s}{|t-s|^{1/2}}\right)^2\right] = m$, so $\sim 1/2$ Hölder regular.

In general, $\mathbb{E}\left[\left(\frac{B_t - B_s}{|t-s|^{1/2}}\right)^p\right] = C_p$.

$$\begin{aligned} [B]_{W^{\alpha,p}} &= \int_{[0,T]^2}\int\left[\frac{|B_t - B_s|}{|t-s|^\alpha}\right]^p \frac{dsdt}{|t-s|} \to \mathbb{E}[[B]_{W^{\alpha,p}}] = \int\int \frac{\mathbb{E}[|B_t - B_s|^p]}{|t-s|^{\alpha p}}\frac{dsdt}{|t-s|} \\ &= \int\int \frac{C_p|t-s|^{p/2}}{|t-s|^{\alpha p + 1}} dtds = C_p \int\int |t-s|^{p\left(\frac{1}{2} - \alpha\right) - 1} dtds \end{aligned}$$

$p\left(\frac{1}{2} - \alpha\right) - 1 > -1 \to \alpha < 1/2$. Thus, $[B]_{W^{\alpha,p}} < \infty$ ($\alpha < 1/2$, $p < \infty$). $W^{\alpha,p} \subset C^{\alpha - \frac{1}{p}}$ ($\alpha > 1/p$ for large enough p). So if $\beta < 1/2$, choose p such that $\alpha = \beta + \frac{1}{p} < 1/2$ so $B \in W^{\alpha,p} \subset C^\beta$.

2.5

We want to move from norm defined by integrals to norm defined by sup.

Theorem 2.1. (Kolmogorov Continuity Criterion) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probabilituy space and (E, d) a complete metric space. Let $X : \Omega \to C([0,1], E)$ be measurable and satisfy, for some $C > 0$, $p > 1$, $\beta \in (0,1)$, $\beta > 1/p$. $\mathbb{E}[d(X_s, X_t)^p] \le C^p|t-s|^{\beta_p}$ $\forall\ t, s \in [0,1]$ $\left(\sup_{s,t\in[0,1]}\left|\left|\frac{d(X_s, X_t)}{(t-s)^p}\right|\right|_{L^p(\Omega)} \le C\right)$.

Then $\forall\ \alpha < \beta - \frac{1}{p}$, $\mathbb{E}\left[\left(\sup_{s,t\in[0,1]}\frac{d(X_s, dX_t)}{|t-s|^\alpha}\right)^p\right]^{1/p} \le MC$ where $M = M(\alpha, \beta, p)$.

Proof. Note that by continuity, it is sufficient to consider only dyadic t, s. Define $K_n = \max_{k=0,\dots,2^n-1} d(X_{k/2^n}, X_{k+1/2^n})$, $||K_n||_p^p \le \sum_{k=0}^{2^n-1} \mathbb{E}[d(X_{k/2^n}, X_{k+1/2^n})^p] \le 2^n C^p 2^{-n\beta_p} = C^p 2^{-n\beta_p - 1}$. Let $D_n = \left(\frac{k}{2^n}\right)_{k=0}^{2^n}$ and $D = \bigcup D_n$.

If we fix $s, t \in D$, they are not necessarily adjacent dyadics, so we need to be careful.

Fix $s, t \in D$ and choose $m \in \mathbb{N}$ such that $2^{-m-1} < t - s \le 2^{-m}$ and get the dyadic expansion of t and s. We can write $s = \frac{k}{2^m} + \sum_{n=m+1}^{M} \frac{\delta_n}{2^n}$, $\delta_n \in \{0,1\}$ and $t = \frac{k+2}{2^m} - \sum_{n=m+1}^{M} \frac{\epsilon_n}{2^n}$, $\epsilon_n \in \{0,1\}$ and M is such that $s, t \in D_M$.

Then

$$\begin{aligned}
\frac{d(X_s, X_t)}{|t-s|^\alpha} &\le 2^{(m+1)\alpha}\left(d(X_{k/2^m}, X_{k+2/2^m}) + 2\sum_{n=m+1}^{\infty} K_n\right) \le 2^{(m+1)\alpha}\cdot 2\sum_{n=m}^{\infty} K_n \\
J_m &= \sup_{s,t\in D: 2^{-m-1}<|t-s|\le 2^{-m}} \frac{d(X_s, X_t)}{|t-s|^\alpha} \le 2^{(m+1)\alpha}\cdot 2\sum_{n=m}^{\infty} K_n \\
||J_m||_{L^p} &\le 2^{m\alpha}2^{(1+\alpha)}\sum_{n=m}^{\infty} C2^{-m(\beta-1/p)} \le \frac{2^{1+\alpha}}{1-2^{-(\beta-1/p)}} C2^{-m(\beta-1/p-\alpha)} \quad \text{(summable)}
\end{aligned}$$

Add up all scales to get all $s, t \in D$, so $||[X]_\alpha||_{L^p} = ||\sup_m J_m||_{L^p} \le \sum_{m=0}^{\infty} ||J_m||_{L^p} \le MC$ □

2.6 Tensor Algebra

$T(V) = \mathbb{R} \oplus V \oplus V\otimes V \oplus V^{\otimes 3} \oplus \cdots = \bigoplus_{n=0}^{\infty} V^{\otimes n}$.

Truncated: $T^{(2)}(V) = \mathbb{R}\oplus V \oplus V\otimes V$. That is, if $(a+V+A)\otimes(b+w+B) = ab + (av+bw) + (v\otimes w + aB + bA)$, i.e. the three tensor vanishes.

$T_1^{(2)} = \{1 + v + A\}$ is a group. $(1+v+A)^{-1} = 1 - v - A + v\otimes v$

Let $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}^\alpha$, then $\mathbf{X}_t = 1 + X_{0,t} + \mathbb{X}_{0,t} \in T_1^{(2)}$ path living in tensor algebra.

Chen's relations become $\mathbf{X}_t \otimes (\mathbf{X}_s)^{-1} = 1 + \delta X_{s,t} + \mathbb{X}_{s,t}$

Suppose $X : [0,T] \to V$, $\mathbb{X} : \Delta_T^2 \to V\otimes V$ satisfies Chen's relations. Define $\mathbf{X}_{s,t} = 1 + \delta X_{s,t} + \mathbb{X}_{s,t}$. Then if $s < t < u$,

$$\begin{aligned}
\mathbf{X}_{s,t}\otimes \mathbf{X}_{t,u} &= 1 + \delta X_{s,t} + \delta X_{t,u} + \mathbb{X}_{s,t} + \mathbb{X}_{t,u} + \delta X_{st}\otimes \delta X_{tu} \\
&= 1 + \delta X_{s,u} + \mathbb{X}_{s,u} \quad \text{(By Chen's relations)} \\
&= \mathbf{X}_{s,u}
\end{aligned}$$

Then taking $\mathbf{X}_t = \mathbf{X}_{0,t}$, $\mathbf{X}_{0,s}\otimes \mathbf{X}_{s,t} = \mathbf{X}_{0,t} \to \mathbf{X}_{s,t} = \mathbf{X}_s^{-1}\otimes \mathbf{X}_t$.

We want a norm on this group. Define $N(X) = \max\{|v|, \sqrt{2|A|}\}$.

Lemma 2.1. (N is additive)

Proof.

$$\begin{aligned}
N(X\otimes Y) &= \max\{|v+w|, \sqrt{2|A+B+v\otimes w|}\}. \\
|v+w| &\le |v| + |w| \le N(X) + N(Y) \\
\sqrt{2|A+B+v\otimes w|} &\le \sqrt{2|A| + 2|B| + 2|v||w|} \le \sqrt{N(X)^2 + N(Y)^2 + N(X)N(Y)} \\
&= \sqrt{(N(X)+N(Y))^2} = N(X) + N(Y)
\end{aligned}$$

□

Define $d(X,Y) = \frac{1}{2}\left[N(X^{-1}\otimes Y) + N(Y^{-1}\otimes X)\right]$ (now also symmetric) and $(T_1^{(2)}(V), d)$ is a complete metric space.

Theorem 2.2. $(X, \mathbb{X}) \in \mathscr{C}^\alpha([0,T], V) \Leftrightarrow \mathbf{X}_{s,t} = 1 + \delta X_{s,t} + \mathbb{X}_{s,t} \in T_1^{(2)}(V)$ satisfies,

1. $\mathbf{X}_{su} = \mathbf{X}_{st}\otimes \mathbf{X}_{tu}$ $\forall\ s < t < u$
2. $(\mathbf{X}_{0,t})_{t\in[0,T]} \in C^\alpha([0,T], (T_1^{(2)}(V), d)$

Let B be Brownian motion. Then $\mathbb{E}[|B_t - B_s|^p] = C_p|t-s|^{p/2}$ $\forall\ p \ge 1 \to$ KCC $B \in C^\alpha$ $\forall\ \alpha < 1/2 - 1/p$.

We now write our motivating RDE as follows

$$dY = f(Y)d\mathbf{X}$$

where $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}^\alpha$.

We consider the example of the Brownian motion. We have previously show that $B \in \mathscr{C}^\alpha$ $\forall\ \alpha < 1/2$. Heuristically, the lift we define

$$\mathbb{B}_{s,t}^{i,j} = \int_s^t (B_r^i - B_s^i)dB_r^j$$

Then, we must consider 2 cases:

(i) $i = j$ That is, we would like to understand the iterated integral of the form

$$\int_s^t B_r dB_r = \lim_{|P|\to 0} \sum_{i=0}^{N-1} B_{t_i}(B_{t_{i+1}} - B_{t_i}) = \lim_{|P|\to 0} I_P$$

We would like the analogue of xdx understood as $d\left(\frac{x^2}{2}\right)$. Then

$$I_P = \sum_{i=0}^{N} \left[\frac{B_{t_{i+1}}^2}{2} - \frac{B_{t_i}^2}{2} - \frac{1}{2}|B_{t_{i+1}} - B_{t_i}|^2 \right] = \frac{B_t^2 - B_s^2}{2} - \frac{1}{2}\sum_{i=0}^{N-1}(B_{t_{i+1}} - B_{t_i})^2$$

Define $Q_P = \sum_{i=0}^{N-1}(B_{t_{i+1}} - B_{t_i})^2$.

Lemma 2.2. $Q_P \overset{L^2(\Omega)}{\to} t - s$ as $|P| \to 0$.

Proof.

$$\begin{aligned}
Q_P - (t-s) &= \sum_{i=0}^{N-1} \left[(B_{t_{i+1}} - B_{t_i})^2 - (t_{i+1} - t_i)\right] = \sum_{i=0}^{N-1} M_{t_i,t_{i+1}} \\
\Rightarrow \mathbb{E}[M_{s,t}] &= 0 \quad \text{(by properties of BM)} \\
\mathbb{E}[M_{s,t}^2] &= C(t-s)^2 \quad \text{(Young's)} \\
\mathbb{E}[|Q_P - (t-s)|^2] &= \sum_{i=0}^{N-1} \mathbb{E}[(M_{t_{i+1},t_i})^2] \quad \text{(off-diagonal terms vanish since } M_{t_i,t_{i+1}} \text{ independent)} \\
&= C\sum_{i=0}^{N-1}(t_{i+1} - t_i)^2 \leq C(t-s)|P| \to 0.
\end{aligned}$$

□

So,

$$\begin{aligned}
\int_s^t (B_r - B_s)dB_r &= \int_s^t B_r dB_r - B_s(B_t - B_s) \\
&= \frac{B_t^2}{2} - \frac{B_s^2}{2} - \frac{1}{2}(t-s) - B_s B_t + B_s^2 \\
&= \frac{(B_t - B_s)^2}{2} - \frac{t-s}{2}.
\end{aligned}$$

(ii) $i \neq j$ Let X, Y independent BM. Then we want to understand the iterated integral $\int_s^t X_r dY_r$. First let's consider $\xi(f) = \int_0^\infty f_r dY_r$ for some deterministic path, $f : [0,\infty) \to \mathbb{R}$.

First, consider a step function $f_r = \sum_{i=1}^{M} a_i \mathbf{1}_{[s_i,t_i]}$ where $s_1 < t_1 < s_2 < t_2 < \dots$. Then

$$\xi(f) = \sum_{i=1}^{M} a_i^2 (Y_{t_i} - Y_{s_i}) \sim \mathcal{N}(0, ||f||_{L^2}^2)$$

since

$$\mathbb{E}[\xi(f)^2] = \sum_{i=1}^{M} a_i^2 \mathbb{E}[(Y_{t_i} - Y_{s_i})^2] = \sum_{i=1}^{M} a_i^2 (t_i - s_i) = \int_0^\infty f_r^2 \, dr = ||f||_{L^2}^2$$

Note: ξ extends as an isometry from $\text{Ł}^2((0,\infty))$ to $L^2(\Omega)$.

Considering our original integral, $\int_s^t X_r dY_r$ since X, Y are independent, changes in X do not change the behavior of Y. Let's look at $(\Omega, \mathbb{P}(\cdot|\mathcal{F}_X))$ so we treat X as "deterministic" or information that we already know and take $\xi_X : L^2((0,\infty)) \to L^2(\Omega)$. We can think of $\Omega = \Omega_X \times \Omega_Y$ and $X(\omega_1, \omega_2) = X(\omega_1)$ and $Y(\omega_1, \omega_2) = Y(\omega_2)$. Then

$$\int_s^t X_r dY_r = \xi_X(\mathbf{1}_{[s,t]} X(\omega_1))(\omega_2) \sim \mathcal{N}\left(0, \int_s^t |X_r(\omega_1)|^2 dr\right)$$

In particular,

$$\begin{aligned}
\mathbb{E}\left[\left|\int_s^t X_r - X_s \, dY_r\right|^p \middle| \mathcal{F}_X\right] &= C_p \mathbb{E}\left[\int_s^t |X_r - X_s|^2 \, dr\right]^{p/2} \\
&\leq C_p \mathbb{E}\left[\int_s^t |X_r - X_s|^p \, dr \, (t-s)^{p/2-1}\right] \\
&= C_p' \int_s^t (r-s)^{p/2} dr \, (t-s)^{p/2-1} = C_p'(t-s)^p
\end{aligned}$$

For Brownian motion, we say that we have moments of all orders and

$$\mathbb{E}[|\mathbb{B}_{s,t}|^p] \leq c_p|t-s|^{p/2} \ \forall\ 0 \leq s \leq t \leq T$$

so,

$$\mathbb{E}[d(\mathbf{B}_s, \mathbf{B}_t)^p] \leq C_p|t-s|^{p/2} \ \forall\ 0 \leq s \leq t \leq T.$$

So, by Kolmogorov Continuity Criterion, $\forall\ \alpha < 1/2$, $\mathbf{B} \in C^\alpha([0,T], T_1^{(2)}(\mathbb{R}^m)) \Leftrightarrow (B, \mathbb{B}) \in \mathscr{C}^\alpha([0,T])$. Choose $\alpha < 1/2$ and then choose p large enough such that $\alpha + \frac{1}{p} < \frac{1}{2}$.

Note: B is not a geometric rough path, we cannot take smooth approximations and expect to converge.

Example 2.3. Let $m = 1$, $\frac{1}{3} < \alpha \leq \frac{1}{2}$, $X \in C^\alpha$. Define $\mathbb{X}_{s,t} = \frac{1}{2}(X_t - X_s)^2$ (in this case this is the only way to lift to a geometric rough path). Consider our RDE, $dY = f(Y)d\mathbf{X}$ and we want to know that $Y_t - Y_s = \int_s^t f(Y_r)d\mathbf{X}_r$ means.

$$Y_s - Y_t \approx f(Y_s)(X_t - X_s) + [(f \cdot \nabla)f](Y_s) : \mathbb{X}_{s,t} + o(|t-s|) = A_{s,t} + o(|t-s|)$$

3 Rough Integration

The Sewing Lemma is how we make this new theory of integration rigorous. It is as follows:

Lemma 3.1. (Sewing Integration Lemma) Fix $0 < \alpha \leq 1 < \gamma$. Then $\exists$ linear maps $\mathcal{I} : C(\Delta_T^2) \to C(\Delta_T^1)$ and $\mathcal{R} : C(\Delta_T^2) \to C(\Delta_T^2)$ such that $A \in C(\Delta_T^2)$. (Let $I_t = \mathcal{I}(A)_t$ and $R_{s,t} = \mathcal{R}(S)_{s,t}$)

1. $I_0 = 0$ and $\delta I_{s,t} = A_{s,t} + R_{s,t}$ (Therefore, we need only construct $\mathcal{I}$ or $\mathcal{R}$)
2. $$\sup_{(s,t)\in\Delta_T^2} \frac{|R_{s,t}|}{|t-s|^\gamma} \lesssim_\gamma \sup_{(s,t,u)\in\Delta_T^3} \frac{|\delta A_{s,t,u}|}{|u-s|^\gamma}$$
3. $$\sup_{(s,t)\in\Delta_T^2} \frac{|\delta I_{s,t}|}{|t-s|^\alpha} \lesssim_{\alpha,\gamma} \sup_{(s,t)\in\Delta_T^2} \frac{|A_{s,t}|}{|t-s|^\alpha} + T^{\gamma-\alpha} \sup_{(s,t,u)\in\Delta_T^3} \frac{|\delta A_{s,t,u}|}{|u-s|^\gamma}$$

Remark 3.1.

- $\delta A_{s,t,u}$ = "how not additive A is" $\therefore$ the remainder is determined by how not additive A is. In fact taking δ of both sides of (i), we get $0 = \delta A + \delta R \to \delta R = -\delta A$
- $$\delta I_{s,t} = \int_s^t A_{r,r+dr} = \lim_{|P|\to 0} \sum_{[u,v]\in P} A_{u,v}$$
  $$\delta I_{s,t} = \sum_{[u,v]\in P} \delta I_{u,v} = \underbrace{\sum_{[u,v]} A_{u,v}}_{\text{converges to something}} + \underbrace{\sum_{[u,v]} R_{u,v}}_{\text{converges to 0}}$$

Example 3.1. $f \in C^\alpha$, $g \in C^\beta$, $s < t < u$, $A_{s,t} = f_s \delta g_{s,t}$.

$$\begin{aligned}
\delta A_{s,t,u} &= f_s(g_u - g_s) - f_s(g_t - g_s) - f_t(g_u - g_t) \\
&= f_s(g_u - g_s - g_t + g_s) - f_t(g_u - g_t) \\
&= (g_u - g_t)(f_s - f_t) = -\delta f_{s,t}\delta g_{t,u} \\
|\delta A_{s,t,u}| &\lesssim [f]_\alpha [g]_\beta |u-s|^{\alpha+\beta}
\end{aligned}$$

Can take $\gamma = \alpha + \beta$ for $\alpha + \beta > 1 \to$ criterion for Young's integration.

Lemma 3.2. Assume $\gamma > 1$, $A : \Delta_T^2 \to W$, $[\delta A]_\gamma = \sup_{s<t<u} \frac{|\delta A_{s,t,u}|}{|u-s|^\gamma} < \infty$ where $\delta A_{s,t,u} = A_{s,u} - A_{s,t} - A_{t,u}$.

Define $\mathcal{I}(A)_{s,t} = \sum_{[u,v]\in P} A_{u,v}$ where P partition of $[s,t]$. Then

$$M(A) = \sup_P |A_{s,t} - \mathcal{I}_P(A)_{s,t}| \lesssim_\gamma [\delta A]_\gamma |t-s|^\gamma$$

Proof. Let P be a partition of $[s,t]$ and let $\#P$ = number of subintervals in P. $\exists v \in P\backslash\{s,t\}$ s.t. if $v_- < v < v_+$ are adjacent points in P, then $|v_+ - v_-| \leq C|t-s|$. If not, then

$$2|t-s| \geq \sum_{v\in P\backslash\{s,t\}} |v_+ - v_-| > C|t-s| \cdot \underbrace{(\#P - 1)}_{\#\text{ of points in } P\backslash\{s,t\}}$$

which yields a contradiction if $C = \frac{2}{\#P-1}$.

Let $\tilde{P} = P\backslash\{v\}$. Then

$$|\mathcal{I}_{\tilde{P}}(A)_{s,t} - \mathcal{I}(A)_{s,t}| = |A_{v_-,v_+} - A_{v_-,v} - A_{v,v_+}| = |\delta A_{v_-,v,v_+}| \leq [\delta A]_\gamma |v_+ - v_-|^\gamma \leq \frac{2^\gamma}{(\#P-1)^\gamma}[\delta A]_\gamma |t-s|^\gamma.$$

Proceeding inductively until reach trivial partition.

$$|A_{s,t} - \mathcal{I}_P(A)_{s,t}| \leq 2^\gamma [\delta A]_\gamma |t-s|^\gamma \sum_{k=1}^{\#P-1} \frac{1}{k^\gamma} = 2^\gamma \zeta(\gamma)[\delta A]_\gamma |t-s|^\gamma$$

□

Proof of Sewing Lemma

Proof. Let P and $\tilde{P}$ be two partitions of $[s,t]$. Assume WLOG that $P \subset \tilde{P}$. Then

$$|\mathcal{I}_P(A)_{st} - \mathcal{I}_{\tilde{P}}(A)_{st}| = \sum_{[u,v]\in P} |A_{uv} - \mathcal{I}_P(A)_{uv}| \le \sum_{[u,v]\in P} M(A)_{uv} \lesssim_\gamma [\delta A]_\gamma \sum_{[u,v]\in P} |v-u|^\gamma \le [\delta A]_\gamma |P|^{\gamma-1}|t-s|$$

so sequence $\{\mathcal{I}_P\}$ is Cauchy. Thus, $\exists$ a limit $\lim_{|P|\to 0} \mathcal{I}_P(A)_{st} = \mathcal{I}(A)_{st}$.

Taking $|\tilde{P}| \to 0$,

$$|\mathcal{I}_P(A)_{st} - \mathcal{I}(A)_{st}| \lesssim_\gamma [\delta A]_\gamma |t-s||P|^{\gamma-1}$$

Taking the coarsest partition $\mathring{P} = \{s,t\}$,

$$|A_{st} - \mathcal{I}(A)_{st}| \lesssim_\gamma [\delta A]_\gamma |t-s|^\gamma$$

That is $\delta\mathcal{I}(A) = 0$ so if $\mathcal{R}(A)_{st} = \mathcal{I}(A)_{st} - A_{st} \Rightarrow \delta R(A)_{st} = -\delta A_{st}$.

So, we get $[\mathcal{R}(A)]_\gamma \lesssim_\gamma [\delta A]_\gamma$

Claim: If $s < t < u$, then $\mathcal{I}(A)_{su} - \mathcal{I}(A)_{st} + \mathcal{I}(A)_{tu}$ $(\delta I(A)_{stu} = 0)$.

Let P be a partition of $[s,u]$ where $t \in P$ as an interior point. Then $P = P_1 \cup P_2$ where P_1 is a partition of $[s,t]$ and P_2 is a partition of $[t,u]$. Then

$$\mathcal{I}(A)_{su} = \mathcal{I}_{P_1}(A)_{st} + \mathcal{I}_{P_2}(A)_{tu}$$

Then $|P_1|, |P_2| \to 0 \Leftrightarrow |P| \to 0$, so $I(A)_t := \mathcal{I}(A)_{0,t} \Rightarrow \mathcal{I}(A)_{s,t} = \delta I(A)_{st}$

[Note: Heuristically, $I(A)_t = \int_0^t A_{r,r+dr}$ and $\delta I(A)_{st} = \int_s^t A_{r,r+dr}$]

Regularity of $\mathcal{I}(A)_t$. We have that $\delta\mathcal{I}(A)_{st} = A_{st} + \mathcal{R}(A)_{st}$. Then

$$\begin{aligned}
\frac{|\delta\mathcal{I}(A)_{st}|}{|t-s|^\alpha|} &\le \frac{|A_{st}|}{|t-s|^\alpha} + \frac{|\mathcal{R}(A)_{st}|}{|t-s|^\gamma}|t-s|^\gamma - \alpha \\
&\le [A]_\alpha + T^{\gamma-\alpha}[\mathcal{R}(A)])_\gamma \\
&\lesssim_\gamma [A]_\alpha + T^{\gamma-\alpha}[\delta A]_\gamma
\end{aligned}$$

□

Example 3.2.

$$\begin{aligned}
A_{st} &= f \otimes \delta g_{st} \\
\Rightarrow \delta A_{stu} &= -\delta f_{st} \otimes \delta g_{tu} \\
\mathcal{R}(A)_{st} &= \int_s^t f_r \delta g_r - f_s \delta g_{st} = \int_s^t [f_r - f_s] \otimes \delta g_r \\
\therefore \delta R &= -\delta A = \delta f_{st} \otimes \delta g_{tu}
\end{aligned}$$

Example 3.3. $(X, \mathbb{X}) \in \mathscr{C}^\alpha$, $\frac{1}{3} < \alpha \le \frac{1}{2}$, f smooth.

$$\begin{aligned}
\int_s^t f(X_r)dX_r &= f(X_s)\delta X_{st} + \int_s^t [f(X_r) - f(X_s)]dX_r \\
&= \underbrace{f(X_s)\delta X_{st} + Df(X_s) \underbrace{\int_s^t (X_r - X_s) \otimes dX_r}_{\mathbb{X}_{st}}}_{A_{st}} + \mathcal{R}
\end{aligned}$$

$$A_{st} = f(X_s)\delta X_{st} + \underbrace{Df(X_s) : \mathbb{X}_{s,t}}_{\text{contraction of tensors: multiply and take trace}}$$

$\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}^\alpha([0,T], V)$, $\frac{1}{3} < \alpha \le \frac{1}{2}$, $f : V \to V$ at least $C^{1,1}$. We want to makes sense of $\int_s^t f(X_r)d\mathbf{X}_r \approx f^i(X_s)\delta X^i_{s,t} + D_{x_i} f^j(X_s) : \mathbb{X}^{i,j}_{s,t} = A_{s,t}$.

[Note: This is not quite the same integral from the RDE $dY_r = f(Y_r)d\mathbf{X}_r$]

$$\begin{aligned}
A_{s,t,u} &= A_{su} - A_{st} - A_{tu} \\
&= -[f(X_t) - f(X_s)] \cdot \delta X_{s,t} + Df(X_s) : \mathbb{X}_{s,u} - Df(X_s) : \mathbb{X}_{s,t} - Df(X_t) : \mathbb{X}_{t,u} \\
&= -\underbrace{[f(X_t) - f(X_s)]}_{\text{like } \delta X_{s,t}} \cdot \delta X_{s,t} + Df(X_s) : \underbrace{[\mathbb{X}_{tu} + \delta X_{st} \otimes \delta X_{tu}]}_{\text{from Chen's relations}} - Df(X_t) : \mathbb{X}_{t,u} \\
&= -[f(X_t) - f(X_s) - Df(X_s) \cdot \delta X_{s,t}] \cdot \delta X_{t,u} - [Df(X_t) - Df(X_s)] : \mathbb{X}_{t,u} \\
&\lesssim ||\nabla^2 f||_\infty ([X]^3_\alpha + [X]_\alpha [\mathbb{X}]_{2\alpha})|u-s|^{3\alpha}
\end{aligned}$$

Thus, we can apply the sewing lemma since $3\alpha > 1$. So,

$$\int_s^t f(X_r)d\mathbf{X}_r = \lim_{|P|\to 0} \sum_{[u,v]\in P} [f(X_u)\delta X_{u,v} + Df(X_u) : \mathbb{X}_{u,v}]$$

exists and

$$\frac{\left|\int_s^t f(X_r)d\mathbf{X}_r\right|}{|t-s|^\alpha} \lesssim ||f||_\infty [X]_\alpha + T^\alpha ||Df||_\infty [\mathbb{X}]_{2\alpha}$$

and

$$\left|\int_s^t f(X_r)dX_r - f(X_s)\delta X_{s,t} - Df(X_s) : \mathbb{X}_{s,t}\right| \lesssim |t-s|^{3\alpha}$$

Can we say anything about the continuity of the map $\mathbf{X} \mapsto \int_0^\cdot f(X_r)d\mathbf{X}_r$? (Stability)

Instead of directly comparing integrals, since $\mathscr{C}^\alpha$ is not a linear space, we compare germs A^1 and A^2 which give us estimates on integrals. Let $A_{s,t} = A^1_{s,t} - A^2_{s,t}$. Then

$$\frac{|\delta A_{s,t,u}|}{|u-s|^{3\alpha}} \lesssim \underbrace{[X^1 - X^2]_\alpha + [\mathbb{X}^1 - \mathbb{X}^2]_{2\alpha}}_{\text{rough path metric}} \ \forall\ \mathbf{X}^1, \mathbf{X}^2 \text{ such that } |||\mathbf{X}^1||| + |||\mathbf{X}^2||| < R$$

That is, locally Lipschitz with respect to this metric.

$dY_r = f(Y_r)d\mathbf{X}_r \to$ "increments of Y up to some multiplicative factor like increments of $\mathbf{X}$"

Definition 3.1. Fix $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}^\alpha([0,T], V)$. $(Y, Y') \in \mathscr{D}^\alpha_{\mathbf{X}}([0,T], W)$ is called an $\mathbf{X}-$controlled rough path if $Y \in C^\alpha([0,T], W)$, $Y' \in C^\alpha([0,T], W \otimes V)$, and $R^Y_{s,t} := \delta Y_{s,t} - Y'_s \delta X_{s,t}$ satisfied $\sup_{t,s} \frac{|R^Y_{s,t}|}{|t-s|^{2\alpha}} < \infty$. That is $\delta Y_{s,t} = Y'_s \delta X_{s,t} + R^Y_{s,t}$.

Example 3.4. $Y_t = f(X_t)$, f smooth.

$$Y_t - Y_s = f(X_t) - f(X_s) = Df(X_s)\delta X_{s,t} + R^Y_{s,t}$$

where $R^Y_{s,t} \lesssim |t-s|^{2\alpha}$ so $(f(X), Df(X)) \in \mathscr{D}^\alpha_{\mathbf{X}}$.

For fixed $\mathbf{X}$, $\mathscr{D}^\alpha_{\mathbf{X}}$ is a linear space and a Banach space with

$$||(Y, Y')||_\alpha = |Y_0| + |Y'_0| + [(Y, Y')]_\alpha$$

where

$$[(Y, Y')]_\alpha = [Y]_\alpha + [R^Y]_{2\alpha}$$

[Note:

$$[Y]_\alpha \le ||Y'||_\infty [X]_\alpha + [R^Y]_{2\alpha} T^\alpha \le (|Y'_0| + [Y'_0]_\alpha T^\alpha)[X]_\alpha + [R^Y]_{2\alpha} T$$

]

$\mathscr{C}^\alpha \ltimes \mathscr{D}^\alpha = \{(\mathbf{X}, Y) : \mathbf{X} \in \mathscr{C}^\alpha,\ Y \in \mathscr{D}^\alpha_{\mathbf{X}}\}$

We can define a kind of "metric". Given $(Y, Y') \in \mathscr{D}^\alpha_{\mathbf{X}}$ and $(\tilde{Y}, \tilde{Y}') \in \mathscr{D}^\alpha_{\tilde{\mathbf{X}}}$ define

$$d((Y, Y'), (\tilde{Y}, \tilde{Y}')) = [Y' - \tilde{Y}']_\alpha + [R^Y - R^{\tilde{Y}}]_{2\alpha} + |Y_0 \tilde{Y}_0| + |Y'_0 - \tilde{Y}'_0|$$

[Note: $d((Y, Y'), (\tilde{Y}, \tilde{Y}') = 0$ does not imply that (Y, Y') and $(\tilde{Y}, \tilde{Y}')$ are not the same objects since they live in different spaces.]

We are concerned with the integration of $Y \in \mathscr{D}^\alpha_{\mathbf{X}}$ against $\mathbf{X} \in \mathscr{C}^\alpha$.

$$\begin{aligned}\int_s^t Y_r d\mathbf{X}_r &\approx Y_s \delta X_{st} + \int_s^t (Y_r - Y_s) dX_r \\ &= Y_s \delta X_{s,t} + Y'_s \underbrace{\int_s^t \delta X_{sr} dX_r}_{\mathbb{X}_{s,t}}\end{aligned}$$

Theorem 3.1.

$$\int_s^t Y_r d\mathbf{X}_r = \lim_{|P| \to 0} \sum_{[u,v] \in P} [Y_u \delta X_{uv} + Y'_u \mathbb{X}_{uv}]$$

and if $Z_t = \int_0^t Y_r d\mathbf{X}_r$ and $Z'_t = Y_t$ then $(Z, Z') \in \mathscr{D}^\alpha_{\mathbf{X}}([0,T], W \otimes V)$ with estimates.

Proof. Define $A_{s,t} = Y_s \delta X_{s,t} + Y'_s \mathbb{X}_{s,t}$. Then to apply sewing lemma, we need to

$$|A_{s,t}| \le (||Y||_\infty [X]_\alpha + ||Y'||_\infty [\mathbb{X}]_{2\alpha} T^\alpha)|t-s|^\alpha$$

$$\begin{aligned}\delta A_{stu} &= Y_s \delta X_{su} - Y_s \delta X_{st} - Y_t \delta X_{tu} + Y'_s \mathbb{X}_{su} - Y'_s \mathbb{X}_{st} - Y'_t \mathbb{X}_{tu} \\ &= -Y_s(\delta X_{su} - \delta X_{st}) - Y_t(\delta X_{tu}) + Y'_s(\underbrace{\ \mathbb{X}_{su} - \mathbb{X}_{st}\ }_{\mathbb{X}_{tu} + \delta X_{st} \otimes \delta X_{tu}}) - Y'_t \mathbb{X}_{tu} \\ &= -\delta Y_{st} \delta X_{tu} = \delta Y'_{st} \mathbb{X}_{tu|} + Y'_s(\delta X_{st} \otimes \delta X_{tu}) \\ &= -(\underbrace{\delta Y_{st} - Y'_s \delta X_{st}}_{R^Y_{st}})\delta X_{tu} - (Y'_t - Y'_s)\mathbb{X}_{tu} \\ &= -R^Y_{st} \delta X_{tu} - \delta Y'_{st} \mathbb{X}_{tu}\end{aligned}$$

So,

$$[\delta A]_{3\alpha} \leq [R^Y]_{2\alpha}[X]_\alpha + [Y'][\mathbb{X}]_{2\alpha}$$

$3\alpha > 1$ so we can apply the sewing lemma. That is $\mathcal{I}(A)_{st} = \int_s^t Y_r d\mathbf{X}_r$ exists,

$$\mathcal{R}(A)_{st} = \mathcal{I}(A)_{st} - A_{st} = \int_s^t Y_r d\mathbf{X}_r - Y_s \delta X_{st} - Y' \mathbb{X}_{st}$$

$[\mathcal{R}(A)]_{3\alpha} \lesssim [\delta A]_{3\alpha}$,

$$\frac{1}{|t-s|^{3\alpha}} \left| \int_s^t \underbrace{\int_s^t Y_r dX_r}_{Z_t - Z_s} - Y_s \delta X_{st} - Y_s' \mathbb{X}_{st} \right| \lesssim_\alpha ([X]_\alpha + [\mathbb{X}]_{2\alpha})([(Y, Y')]_\alpha)$$

$\delta Z_{st} = Y_s \delta X_{st} + \underbrace{\overbrace{Y_s' \mathbb{X}_{st}}^{|t-s|^{2\alpha}} + \overbrace{\mathcal{R}(A)_{st}}^{|t-s|^{3\alpha}}}_{R^Z_{st}}$ so we get a higher order estimation/expansion + remainder that is more regular. This makes sense intuitively that as you integrate you should gain regularity. $[Z']_\alpha = [Y]_\alpha$ and $[R^Z]_{2\alpha} \leq ||Y'||_\infty [\mathbb{X}]_{2\alpha} + ([X]_\alpha + [\mathbb{X}]_{2\alpha})[(Y, Y')]_\alpha T^\alpha$,

$$[(Z, Z')] \lesssim_{\alpha, T} (|Y_0'| + [(Y, Y')])([X]_\alpha + [\mathbb{X}]_{2\alpha})$$

□

$(X, \mathbb{X}) \in \mathscr{C}^\alpha([0, T], V)$, $(Y, Y') \in \mathscr{D}^\alpha_{\mathbf{X}}$.

$$\delta Z_{st} = \int_s^t Y_r \, d\mathbf{X}_r = \lim_{|P| \to 0} \sum_{[u,v] \in P} [Y_u \delta X_{u,v} + Y_u' \mathbb{X}_{uv}]$$

so, $(Z, Z') \in \mathscr{D}^\alpha_{\mathbf{X}}$, $Z' = Y$.

Remark 3.2.

1. We took $Y \in W \to Z \in W \otimes V = \mathscr{L}(V, W)$ "linear maps from V to W", then $Z' \in (W \otimes V) \otimes V$. So how is $Z' = Y$? There exists a canonical inclusion $W \hookrightarrow (W \otimes V) \otimes V$ where $w \mapsto i(w)[v] = w \otimes v$. We could also take $Y \in \tilde{W} \otimes V$, $Y' \in (\tilde{W} \otimes V) \otimes V \Rightarrow Z \in \tilde{W}$, $Z' = Y \in \tilde{W} \otimes V$ and don't have to think about inclusion.
2. Recall if $(X, \mathbb{X}), (X, \tilde{\mathbb{X}}) \in \mathscr{C}^\alpha([0, T], V)$, then $\tilde{\mathbb{X}} = \mathbb{X} + \delta F$ for some $F \in C^{2\alpha}([0, T], V \otimes V)$ by Chen's relations.

   If $(Y, Y') \in \mathscr{D}^\alpha_{\mathbf{X}} \Leftrightarrow (Y, Y') \in \mathscr{D}^\alpha_{\tilde{\mathbf{X}}}$ because they have the same base path. So, we could integrate against $\mathbf{X}$ or $\tilde{\mathbf{X}}$, but the integrals will be different.

$$\begin{aligned} \int_s^t Y_r d\tilde{\mathbf{X}}_r &= \lim_{|P| \to 0} \sum_{[u,v] \in P} [Y_u \delta X_{uv} + Y_u \tilde{\mathbb{X}}_{uv}] \\ &= \lim_{|P| \to 0} \sum_{[u,v] \in P} [Y_u \delta X_{uv} + Y_u' \mathbb{X}_{uv} + Y_u' \delta F_{uv}] \\ &= \int_s^t Y_r d\mathbf{X}_r + \underbrace{\int_s^t Y_r' dF_r}_{\text{correction term}} \end{aligned}$$

   [Note: $\int Y' dF$ makes sense since $Y' \in C^\alpha$ and $F \in C^{2\alpha}$, so we can apply the sewing lemma.]

3.1 Stability Estimates for Rough Integration

$(X, \mathbb{X}), (\tilde{X}, \tilde{\mathbb{X}}) \in \mathscr{C}^\alpha$, $(Y, Y') \in \mathscr{D}^\alpha_{\mathbf{X}}, (\tilde{Y}, \tilde{Y}') \in \mathscr{D}^\alpha_{\tilde{X}}$. Then $Z = \int Y d\mathbf{X}$ and $\tilde{Z} = \int \tilde{Y} d\tilde{\mathbf{X}}$. We'd like to estimate a kind of "distance" between Z and $\tilde{Z}$.

$$[(Z, Z'), (\tilde{Z}, \tilde{Z}')]_\alpha := [Z' - \tilde{Z}']_\alpha + [R^Z - R^{\tilde{Z}}]_{2\alpha}$$

[Note: even if $\mathbf{X} = \tilde{\mathbf{X}}$, $\tilde{Z} = Z + c_1 + c_2 X$ so $\tilde{Z}' = Z' + c_2$, then $[Z' - \tilde{Z}']_\alpha = 0$ and $\delta \tilde{Z}_{st} = \delta Z_{st} + c_2 \delta X_{st} = (Z_s' + c_2)\delta X_{st} + R^Z_{st}$, so $[R^Z - R^{\tilde{Z}}]_{2\alpha} = 0$. The pseudonorm will measure distance of 0, i.e. will not separate]

Theorem 3.2. If $M > 0$ and $[X]_\alpha, [\mathbb{X}]_{2\alpha}, [\tilde{X}]_\alpha, [\tilde{\mathbb{X}}]_{2\alpha}, [(Y, Y')]_\alpha, [(\tilde{Y}, \tilde{Y}')]_\alpha, |Y_0|, |Y_0'| \leq M$. Then

$$[(Z, Z'), (\tilde{Z}, \tilde{Z}')]_\alpha \lesssim_M \rho_\alpha(\mathbf{X}, \tilde{\mathbf{X}}) + |Y_0' - \tilde{Y_0}'| + [(Y, Y'), (\tilde{Y}, \tilde{Y}')]T^\alpha$$

and

$$[Z - \tilde{Z}]_\alpha \lesssim_M |Y_0 - \tilde{Y}_0| + \rho_\alpha(\mathbf{X}, \tilde{\mathbf{X}}) + |Y_0' - \tilde{Y_0}'| + [(Y, Y'), (\tilde{Y}, \tilde{Y}')]T^\alpha$$

Proof. $Z' = Y$, $\tilde{Z}' = \tilde{Y}$

$$\begin{aligned}
[Z' - \tilde{Z}']_\alpha &= [Y - \tilde{Y}]_\alpha \\
&\le ||Y' - \tilde{Y}'||_\infty [X]_\alpha + ||\tilde{Y}'||_\infty [X - \tilde{X}]_\alpha + \underbrace{[R^Y - R^{\tilde{Y}}]_\alpha}_{\le [R^Y - R^{\tilde{Y}}]_{2\alpha} T^\alpha} \\
&\lesssim_M |Y_0' - \tilde{Y}_0'| + [Y' - \tilde{Y}']_\alpha T^\alpha + \rho_\alpha(\mathbf{X}, \tilde{\mathbf{X}}) + [R^Y - R^{\tilde{Y}}]_{2\alpha} T^\alpha \\
&= |Y_0' - \tilde{Y}_0'| + [(Y, Y'), (\tilde{Y}, \tilde{Y}')] T^\alpha + \rho_\alpha(\mathbf{X}, \tilde{\mathbf{X}})
\end{aligned}$$

$$\begin{aligned}
\delta Z_{st} &= Y_s \delta X_{st} + Y_s' \mathbb{X}_{st} + \mathcal{R}(A)_{st} \\
\delta \tilde{Z}_{st} &= \tilde{Y}_s \delta \tilde{X}_{st} + \tilde{Y}_s \tilde{\mathbb{X}}_{st} + \mathcal{R}(\tilde{A})_{st}
\end{aligned}$$

The difference of the first two terms will be as above,

$$[R^Z - R^{\tilde{Z}}]_{2\alpha} \lesssim_M ||Y' - \tilde{Y}'||_\infty + [\mathbb{X} - \tilde{\mathbb{X}}]_{2\alpha} + \underbrace{[\mathcal{R}(A - \tilde{A})]_{2\alpha}}_{\le T^\alpha [\mathcal{R}(A - \tilde{A})]_{3\alpha} \lesssim T^\alpha [\delta A - \delta \tilde{A}]_{3\alpha}}$$

$$\begin{aligned}
\delta A_{stu} &= -R_{st}^Y \delta X_{tu} - \delta Y_{st}' \mathbb{X}_{tu} \\
\delta \tilde{A}_{stu} &= -R_{st}^{\tilde{Y}} \delta \tilde{X}_{tu} - \delta \tilde{Y}_{st}' \tilde{\mathbb{X}}_{tu}
\end{aligned}$$

Taking the difference and proceeding as above. □

3.2 Existence and uniqueness

$$dY_t = f(Y_t) d\mathbf{X}_t, Y_0 = y \in W, \ \mathbf{X} \in \mathscr{C}^\alpha([0,T], V), \ f : W \to W \otimes V$$

$\delta Y_{st} = \int_s^t f(Y_r) d\mathbf{X}_r$ where $f(Y)$ needs to be a controlled rough path for us to make sense of integration

If $(Y, Y') \in \mathscr{D}_{\mathbf{X}}^\alpha([0,T], W)$, then

$$f(Y_t) - f(Y_s) = Df(Y_s)\delta Y_{st} + \underbrace{\int_0^t Df(\tau Y_t + (1-\tau)Y_s) - Df(Y_s) d\tau \delta Y_{st}}_{R_{st}^{fy}} = Df(Y_s) Y_s' \delta X_{st} + Df(Y_s) R_{st}^Y + R_{st}^{fy}$$

$|R_{st}^{fy}| \le ||Df||_\infty |\delta Y_{st}|^2 \Rightarrow (f(Y), f(Y)') \in \mathscr{D}_{\mathbf{X}}^\alpha([0,T], W \otimes V)$ where $f(Y)' = Df(Y)Y'$. Note: we need $Y' \in C^\alpha$ since $Df(Y) \in C^\alpha$ since Df is Lipschitz.

We say $(Y, Y') \in \mathscr{D}_{\mathbf{X}}^\alpha$ solves RDE if $Y_0 = y$ and $\delta Y_{st} = \int_s^t f(Y_r) d\mathbf{X}_r \ \forall \ s, t$. (Y, Y') solves RDE $\Rightarrow Y' = f(Y)$ (looks like an ODE expect prime is with respect to X)

Can we prove existence and uniqueness?

Theorem 3.3. If $f \in C_b^2$, then $\exists$ a solution. If $f \in C_b^3$, then the solution is unique.

We want apriori estimates for $[Y]_\alpha$ (in terms of $\mathbf{X}$):

$$\begin{aligned}
\delta Y_{st} &= f(Y_s) \delta X_{st} + R_{st}^Y \\
&= f(Y_s) \delta X_{st} + f(Y_s)' \mathbb{X}_{st} + \mathcal{R}(A)_{st}
\end{aligned}$$

Note also,

$$\begin{aligned}
A_{st} &= f(Y_s) \delta X_{st} + f(Y_s)' \mathbb{X}_{st} \\
&= -R_{st}^{f(Y)} \delta X_{tu} - \delta f(Y)_{st} \mathbb{X}_{tu}
\end{aligned}$$

Goal: Choose h and estimate $[R^Y]_{2\alpha,h} = \sup \left\{ \frac{|R_{st}^Y|}{|t-s|^{2\alpha}} : |t-s| \le h \right\}$

$$\begin{aligned}
R_{st}^Y &= f(Y_s)' \mathbb{X}_{st} + \mathcal{R}(A)_{st} \\
[R^Y]_{2\alpha,h} &\lesssim ||Y'||_\infty [\mathbb{X}]_{2\alpha} + [\mathcal{R}(A)]_{3\alpha,h} h^\alpha \\
&\lesssim [\mathbb{X}]_{2\alpha} + [\delta A]_{3\alpha,h} h^\alpha \\
&\lesssim [\mathbb{X}]_{2\alpha} + ([R^{f(Y)}]_{2\alpha,h} [X]_\alpha + [f(Y)']_{\alpha,h} [\mathbb{X}]_{2\alpha}) h^\alpha \\
&\le |t-s|^{2\alpha} ([Y]_{\alpha,h}^2 + [R^Y]_{2\alpha,h})
\end{aligned}$$

Take $h^\alpha |||\mathbf{X}||| < 1/2$.

$$[R^Y]_{2\alpha,h} \le \frac{1}{2} [R^Y]_{2\alpha,h} + C(||\mathbb{X}||_{2\alpha} + \frac{1}{2}[Y]_{\alpha,h}^2) + ||\mathbb{X}||_{2\alpha}^{1/2} [Y]_{\alpha,h} \lesssim [\mathbb{X}]_{2\alpha} + [Y]_{\alpha,h}^2$$

Since $\delta Y_{st} = F(Y_s) \delta X_{st} + R_{st}^Y$, then we can close the loop.

$$\begin{aligned}
|\delta Y_{st}| &\lesssim ([X]_\alpha + h^\alpha [R^Y]_{2\alpha,h}) |t-s|^\alpha \\
h^\alpha [Y]_{\alpha,h} &\le h^\alpha [X]_\alpha + h^{2\alpha} ([\mathbb{X}]_{2\alpha} + [Y]_{\alpha,h}^2)
\end{aligned}$$

Note: Suppose $Nh < |t-s| < (N+1)h$. Then

$$\frac{|\delta Y_{st}|}{|t-s|^\alpha} \leq \frac{(N+1)[Y]_{\alpha,h}h^\alpha}{(Nh)^\alpha} \lesssim N^{1-\alpha}[Y]_{\alpha,h}$$

so $[Y]_{\alpha;[0,T]} \lesssim \frac{T^{1-\alpha}}{h^{1-\alpha}}[Y]_{\alpha,h;[0,T]}$

We required that $h^\alpha|||\mathbf{X}|||_\alpha \lesssim 1 \Rightarrow [Y]_{\alpha,h} \lesssim [Y]^2_{\alpha,h}h^\alpha + |||\mathbf{X}|||_\alpha$ We want to specify h is a clever way so we can estimate in terms of X since we are trying to bound $[Y]_\alpha$

Multiplying by h^α and by C

$$\begin{aligned}&\Rightarrow h^\alpha[Y]_{\alpha,h} \leq C(h^\alpha[Y]_{\alpha,h})^2 + Ch^\alpha|||\mathbf{X}|||_\alpha\\ &\Rightarrow \underbrace{Ch^\alpha[Y]_{\alpha,h}}_{\psi_h} \leq (Ch^\alpha[Y]_{\alpha,h})^2 + \underbrace{C^2h^\alpha|||\mathbf{X}|||_\alpha}_{\lambda_h}\\ &\Rightarrow \psi_h \leq \psi_h^2 + \lambda_h\end{aligned}$$

When $h \to 0$, ψ_h gets small and can absorb ψ_h^2. We also know that $\lambda_h \to 0$.

We take $\lambda_h \leq 1/4$ since $\psi_h \leq \frac{1}{2}(2\psi_h^2 + 1/2) = \psi_h^2 + 1/4$ to give actual conditions on ψ_h. So, $\psi_\pm = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \lambda_h}$ then either $\psi_h \geq \psi_+ > 1/2$ or $\psi_h \leq \psi_- \leq 2\lambda_h$ We want to specify λ_h so such a jump cannot occur before λ_h.

Take h_0 such that $\lambda_{h_0} < \frac{1}{100}$. Claim: $\phi_h \leq \psi_- \ \forall\ h \leq h_0$.

If $\lambda_h \leq \frac{1}{100}, \psi_- \leq \frac{1}{50}$. We actually claim $\psi_h \leq 2\lim_{\delta\to 0}\psi_{h-\delta}$ which would rule a jump by a factor of 25.

$$h^\alpha\frac{|Y_t - Y_s|}{|t-s|^\alpha} \leq h^\alpha\left[\frac{|Y_t - Y_s|}{2^\alpha|t-m|^\alpha} + \frac{|Y_m - Y_s|}{2^\alpha|s-m|^\alpha}\right] \leq \left(\frac{h}{2}\right)^\alpha 2[Y]_{\alpha,h/2} \Rightarrow h^\alpha[Y]_{\alpha,h} \leq 2\left(\frac{h}{2}\right)^\alpha[Y]_{\alpha,h/2}$$

$\psi_h \leq 2\psi_{h/2} \leq 2\lim_{\delta\to 0}\psi_{h-\delta}$ since ψ is increasing.

So, $[Y]_{\alpha,h} \lesssim |||\mathbf{X}|||_\alpha$. If

$$h^\alpha|||\mathbf{X}|||_\alpha = c \Rightarrow [Y]_{\alpha;[0,T]} \lesssim_T \frac{1}{h^{1-\alpha}}[Y]_{\alpha,h;[0,T]} \lesssim \frac{1}{h^{1-\alpha}}|||\mathbf{X}|||_\alpha$$

Then, we get existence for RDE for $f \in C^{1,1}$ by Euler scheme.

Uniqueness: $Z_t = y + \int_0^t f(Y_s)d\mathbf{X}_s$ solution to RDE $\Leftrightarrow (Y,Y') \overset{m}{\mapsto} (Z, f(Y))$

proof (sketch): Set up fixed point problem. First, take α' such that $1/3 < \alpha' < \alpha \leq 1/2$. Look for fixed point of m on $\mathscr{D}^\alpha_\mathbf{X}$. (Note if $\mathbf{X} \in \mathscr{C}^\alpha$, then $\mathbf{X} \in \mathscr{C}^{\alpha'}$) Then, we can show that $(Y,Y') \in \mathscr{D}^\alpha_\mathbf{X}$.

$$\frac{|\delta Y_{st}|}{|t-s|^\alpha} = |f(Y_s)|\frac{|\delta X_{st}|}{|t-s|^\alpha} + \frac{|R^Y_{st}|}{|t-s|^{2\alpha'}}|t-s|^{2\alpha'-\alpha}$$

where $1/2\alpha < \alpha' < \alpha$. Next, $B_T = \{(Y,Y') \in \mathscr{D}^{\alpha'}_\mathbf{X} : Y_0 = y, Y'_0 = f(y), [(Y,Y')]_\alpha \leq 1\}$ Need to if T is sufficiently small that $m : B_T \to B_T$ and m is a contraction mapping.

3.3 Itô v.s. Stratonovich integrals

$\frac{1}{3} < \alpha \leq \frac{1}{2}$, $\mathbf{X} \in \mathscr{C}^\alpha$, $f \in C_b^{2,\delta}$, $\delta > \frac{1-2\alpha}{\alpha}$, $dY = f(Y)d\mathbf{X} \Rightarrow \exists!$ solution $(Y, f(Y)) \in \mathcal{D}^\alpha_\mathbf{X}$. Also, stability: $\mathbf{X}, \tilde{\mathbf{X}}, y, \tilde{y} \Rightarrow (Y, f(Y)), (\tilde{Y}, f(\tilde{Y}))$ solutions.

$$[Y - \tilde{Y}]_\alpha + [R^Y - R^{\tilde{Y}}]_{2\alpha} \lesssim |y - \tilde{y}| + \rho_\alpha(\mathbf{X}, \tilde{\mathbf{X}})$$

Recall, a geometric rough path $(X, \mathbb{X}) \in \mathscr{C}^\alpha_g$ is a rough path such that $\mathrm{Sym}(\mathbb{X}_{s,t}) = \frac{1}{2}\delta X_{s,t} \otimes \delta X_{s,t} = \int_s^t (X^i_r - X^i_s)dX^j_r + \int_s^t (X^j_r - X^j_s)dX^i_r$ (integration by parts formula) $\Leftrightarrow\ \exists\ (X^{(n)}) \in C^1$ such that $X^{(n)} \to X$ uniformly and $\mathbb{X}^{(n)}_{s,t} = \int_s^t X^{(n)}_s - X^{(n)}_r) \otimes dX^{(n)}_r \to \mathbb{X}_{s,t}$ uniformly and $\sup_n |||(X^{(n)}, \mathbb{X}^{(n)})|||_\alpha < \infty$.

These properties are not clear, in fact any random approximation will not have the second property.

Ex: $V = \mathbb{R}^2$, $X^{(n)}_t = \frac{\alpha}{n^{1/2}}(\cos(2\pi nt), \sin(2\pi nt))$. Then $X^{(n)} \to 0$ and $\mathbb{X}^{(n)}_{s,t} \to \frac{\alpha^2}{2}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}(t-s)$ but $\mathbf{X} = (0, \mathbb{X}) \in \mathscr{C}^{1/2}_g$.

If $f : \mathbb{R} \to \mathbb{R} \otimes \mathbb{R}^2$. $\dot{Y}^{(n)} = f(Y^{(n)}) \cdot \dot{X}^{(n)}$, then by stability, $Y^{(n)} \to Y$ that solves $dY = f(Y)d\mathbf{X}$. Then $\dot{Y} = \frac{\alpha^2}{2}\{f_1, f_2\}(y) = \frac{\alpha^2}{2}(f_1f_2' - f_2f_1')(y)$.

Brownian Motion: We define $\mathbb{B}^I_{s,t} = \int_s^t (B_r - B_s) \otimes dB_r$ (as L^2 limits of Riemann sums). Let $\mathbf{B}^I = (B, \mathbb{B}^I) \in \mathscr{C}^\alpha$. $dY = f(y)d\mathbf{B}^I \Leftrightarrow dY = f(y)dB$. Note $\mathbf{B}^I \notin \mathscr{C}^\alpha_g$ since

$$\mathrm{Sym}(\mathbb{B}^I_{s,t}) = \frac{1}{2}\left(\mathbb{B}^{I,ij}_{s,t} + \mathbb{B}^{I,ji}_{s,t}\right) = \begin{cases}\frac{1}{2}\delta B^i_{st}\delta B^j_{st} & i \neq j\\ \frac{1}{2}(\delta B^i_{st})^2 - \frac{t-s}{2} & i = j\end{cases}$$

So, fails to be geometric, but by another smooth path.

Define $\mathbb{B}^S_{s,t} = \mathbb{B}^I_{s,t} + \frac{1}{2}(t-s)Id$. Then, $\mathbf{B}^s = (B, \mathbb{B}^S) \in \mathscr{C}^\alpha_g$ since $\frac{1}{2}(t-s)Id$ smooth and we have integration by parts. It turns out that $\mathbb{B}^S_{s,t} = \lim_{|P|\to 0}\sum_{i=1}^N (B_{(t_i+t_{i+1})/2} - B_s) \otimes \delta B_{t_i,t_{i+1}}$ (Stratonovich). So, we can see that the intermediate point does matter, unlike Riemann integration.

$$dY = f(Y)d\mathbf{B}^S_t \Leftrightarrow dY = f(Y) \circ d\mathbf{B} = f(Y) \cdot d\mathbf{B}^I + \underbrace{\frac{1}{2}f(Y) \cdot Df(Y)dt}_{\text{difference between Itô and Stratonovich}}$$

Y from Itô integral form is a martingale. Y from Stratonovich integral form is not a martingale, $\mathbb{E}[Y_{s,t}] = \frac{t-s}{2}$ but we gain nice calculus facts like integration by parts and the chain rule, and we know that we can approximate by smooth path since geometric rough path.

Wang-Zakai: What is the correct smooth approximation to get $(B, \mathbb{B}) \in \mathscr{C}^\alpha$? If $(B^{(n)})$ is either $B * \rho_{1/n}$ where ρ a smooth mollifier, approximation of identity or if take piecewise linear interpolation of stepsize $1/n$ and $\mathbb{B}^{(n)} = \int \delta B^{(n)} \otimes \delta B^{(n)}$, then $(B^{(n)}, \mathbb{B}^{(n)}) \to (B, \mathbb{B}^S)$ in C^α.

There are other lifts that give geometric rough paths so there are other ways to approximate that will converge to these other geometric rough paths.

4 Gaussian Measure Theory

- on $\mathbb{R}$:
$$\mu(dx) = \begin{cases} \frac{1}{(2\pi)^{1/2}\sigma} \exp\left(-\frac{|x|^2}{2\sigma^2}\right) dx & \sigma > 0 \\ \delta_0 & \sigma = 0 \end{cases}$$
density for Gaussian with mean 0 and variance $\sigma \Leftrightarrow \int |x|^2 d\mu = \sigma^2$.

- on $\mathbb{R}^d$: A Borel probability measure μ is a (centered) Gaussian measure if $\ell^*\mu$ is a centered Gaussian measure on $\mathbb{R}$ $\forall$ $\ell : \mathbb{R}^d \to \mathbb{R}$ linear where $\ell^*\mu(A) = \mu(\ell^{-1}(A))$ refers the pushforward of μ through ℓ. In this case, we cannot speak about a single variance. Instead, we define a covariance matrix,
$$\Sigma_{ij} = \int x_i x_j \mu(dx) = \mathbb{E}[X_i X_j]$$
if $(X_1, \ldots, X_d)$ has distribution μ. Equivalently, for $\ell, \tilde{\ell} \in \mathbb{R}^d$, $\Sigma\ell \cdot \tilde{\ell} = \int (\ell \cdot x)(\tilde{\ell} \cdot x)\mu(dx)$. It turns out that $\Sigma \geq 0$, symmetric, and if Σ is invertible, then
$$\mu(dx) = \frac{1}{(2\pi)^{d/2}\det(\Sigma)^{1/2}} \exp\left(-\frac{1}{2}\Sigma^{-1}x \cdot x\right) dx$$

- on Banach spaces:
  - Brownian Motion, $B(\omega) : [0, \infty) \to \mathbb{R}^d$, $B(\omega) \in C([0, \infty), \mathbb{R}^d)$. Then for $A \in C([0, T], \mathbb{R}^d)$, $\mu(A) = \mathbb{P}[B(\omega) \in A]$ is a Gaussian measure.
  - White noise, $\xi : \mathbb{R}^d \to \mathbb{R}$ random distribution, $\mathbb{E}[\xi(x)\xi(y)] = \delta(x - y) \Leftrightarrow \xi : C_c^\infty(\mathbb{R}^d) \to \mathbb{R}$, $\mathbb{E}[\langle \xi, \varphi \rangle \langle \xi, \psi \rangle] = \int \varphi(x)\psi(x)\, dx$ gives Gaussian measure on some space of distributions.
    $\xi : L^2(\Pi^d) \hookrightarrow L^2(\Omega)$. Let (ξ_k) be a sequence of independent, centered, variance 1, Gaussians on $\mathbb{R}$. That is, some orthonormal basis for a closed subspace of $L^2(\Omega)$. $(e^{ik\cdot x})_{k\in\mathbb{Z}^d}$ orthonormal basis of $L^2(\Pi^d)$. $\xi = \sum_k \xi_k e^{ik\cdot x}$ is not summable with probability 1. However, $\xi \in H^{-s}(\Pi^d) = \left\{T \in \mathcal{D}'(\Pi^d) : \sum \frac{|\hat{\xi}(k)|^2}{(1+|k|^2)^s} < \infty\right\}$ for some $s > 0$. That is $\left(\frac{\hat{\xi}(k)}{(1+|k|^2)^{s/2}}\right) \in \ell^2(\mathbb{Z}^d)$.
    $$\mathbb{E}\left[\sum_k \frac{\xi_k^2}{(1 + |k|^2)^s}\right] = \sum_k \frac{1}{(1 + |k|^2)^s} < \infty$$
    for $s > d/2$. Then $L^2(\Pi^d) \subset H^{-s}(\Pi^d)$, $s > d/2$. Thus, white noise is a Gaussian measure in such $H^{-s}(\Pi^d)$.
    μ on $L^2(\Pi^d)$ gives most (all?) information about μ supported on $H^{-s}(\Pi^d)$, a much larger space.

  Let $\mathcal{B}$ be a separable Banach space, μ a Borel probability measure is Gaussian if $\ell^*\mu$ is Gaussian on $\mathbb{R}$ $\forall$ $\ell \in \mathcal{B}^*$. We can think of mean, $m(\ell) = \int \ell(x)\mu(dx), \sim m \in \mathcal{B}^{**}$. We actually get $m \in B$.

$\mathcal{B}$ separable Banach space, $\mathcal{P}(\mathcal{B})$ = Borel probability measures
Recall, $\mu \in \mathcal{P}(\mathcal{B})$ is a (centered) Gaussian $\Leftrightarrow$ $\ell^*\mu$ is a (centered) Gaussian in $\mathbb{R}$ for all $\ell \in \mathcal{B}^*$.

Remark 4.1. If $\mu, \nu \in \mathcal{P}(\mathcal{B})$ and $\ell^*\mu = \ell^*\nu$ $\forall$ $\ell \in \mathcal{B}^*$, then $\mu = \nu$.

We can define a canonical random variable with measure μ that we can identify with μ.

Let $\Omega = \mathcal{B}$, $\mathcal{F}$ = Borel sets $\mathbb{P} = \mu$. Then $\Omega \ni \omega \mapsto X(\omega) = \omega \in B$ is an $\mathcal{F}-$measurable random variable nad $\mathbb{P}(X \in A) = \mu(A)$ by construction.

Then, μ is a Gaussian measure $\Leftrightarrow$ $\ell(X)$ is a centered Gaussian on $\mathbb{R}$ $\forall$ $\ell \in \mathcal{B}^*$.

Definition 4.1. The *covariance operator* $\Sigma_\mu : \mathcal{B}^* \times \mathcal{B}^* \to \mathbb{R}$ where $(\ell, \tilde{\ell}) \mapsto \int \ell(x)\tilde{\ell}(x)\mu(dx)$. This is well-defined since $\int \ell(x)^2\mu(dx) = \int_\mathbb{R} y^2(\ell^*\mu)(dy) < \infty$ since μ is Gaussian. This operator is finite, bilinear, symmetric, and nonnegative.

Theorem 4.1. (Fernique's theorem) $\exists$ universal constant $\alpha > 0$ such that $\mu(||x|| > t) \leq \exp\left(\frac{-\alpha t^2}{M^2}\right)$ where $M = \int ||x||\mu(dx)$ $\forall$ $t > M$. (In fact, all moments can be estimated in terms of the first moment.)

Proof. Let $0 < \tau < t$. Then
$$\mu(||x|| > t)\mu(||x|| < \tau) = (\mu \otimes \mu)(\{||x|| > t\} \times \{||x|| < \tau\}) = \mu(R_{\pi/4}\{||x|| > t\} \times \{||x|| < \tau\}) \leq \mu(||x|| > \frac{t - \tau}{\sqrt{2}})^2$$

where we use that μ is rotationally invariant to rotate the set $\{||x|| > t\} \times \{||x|| < \tau\}$ by $\pi/4$ and estimate this set in $\{||x|| < \frac{t-\tau}{2}\}$. Let $c = \mu(||x|| < \tau)$. Then by a change of variables,

$$\frac{1}{c}\mu(||x|| > \sqrt{2}s + \tau) \leq \frac{\mu(||x|| > s)}{c^2}$$

Let $s_0 = \tau$ and $s_{n+1} = \sqrt{2}s_n + \tau$ and define $y_n = \frac{1}{c}\mu(||x|| > s_n)$. Then the above inequality says $y_{n+1} \leq y_n^2 \Rightarrow y_n \leq y_0^{2^n} = \left(\frac{1}{c}\mu(||x|| < \tau)\right)^{2^n} = \left(\frac{1-c}{c}\right)^{2^n} \leq 3^{-2^n}$ since we can take c to be as close to 1 as we like since $\mu(||x|| \leq \tau) \to 1$ as $\tau \to \infty$. Thus, choose τ such that $c \geq \frac{3}{4}$.

$$s_n = \frac{(\sqrt{2})^{n+1} - 1}{\sqrt{2} - 1}\tau. \quad s_n \leq C2^{n/2}\tau$$

Now, take $t \leq \tau \to s_n \leq t \leq s_{n+1}$. Then

$$\mu(||x|| > t) \leq \mu(||x|| > s_n) \leq 3^{-2^n} = e^{-\tilde{C}(s_{n+1})^2/\tau^2} \leq e^{-C^2t^2/\tau^2}$$

[So, $\int ||x||\mu(dx) < \infty$. Applying Chebyshev (Markov), $\mu(||x|| \geq \tau) \leq \frac{M}{\tau}$. Take $\tau = 4M$, so $\mu(||x|| > \tau) < 1/4$. So, $c \geq \frac{3}{4}$] □

Corollary 4.1. (1) ∃ constant $||\Sigma_\mu||$ such that $\Sigma_\mu(\ell, \tilde{\ell}) \leq ||\Sigma_\mu|| ||\ell|| ||\tilde{\ell}||$. Also, (2) $\tilde{\Sigma} : \mathcal{B}^* \to \mathcal{B}$ is continuous.

Proof. 1. $\ell(x) \leq ||\ell||_{\mathcal{B}^*}||x||_{\mathcal{B}}$

2. $\tilde{\Sigma}_\mu(\ell) = \underbrace{\int x\ell(x)\mu(dx)}_{\text{Bochner integral}}$ (well-defined b/c $||x\ell(x)|| \leq ||\ell||_{\mathcal{B}^*}||x||^2_{\mathcal{B}}$ and $||x||^2_{\mathcal{B}}$ integrable since x is Gaussian.)

□

4.1 Ito integral

$\mathbf{X} \in \mathscr{C}^\alpha([0,T], V)$, $\frac{1}{3} < \alpha \leq \frac{1}{2}$.

$$\mathrm{Sym}(\mathbb{X}_{st}) = \frac{1}{2}\delta X_{st} \otimes \delta X_{st} + E_{st}$$

where $E_{s,t} \in \mathrm{Sym}(V \otimes V)$. For geometric rough paths, $E_{st} = 0$.

Claim: $E_{st} = \delta\gamma_{st} = \gamma_t - \gamma_s \Leftrightarrow \delta E_{stu} = E_{su} - E_{st} - E_{tu} = 0$

$$\begin{aligned}
\delta E_{stu} &= \mathrm{Sym}(\delta\mathbb{X}_{stu}) - \delta(\frac{1}{2}\delta X \otimes \delta X)_{stu} \\
\delta\mathbb{X}_{stu} &= \delta X_{st} \otimes \delta X_{tu} \quad \text{(Chen's relations)} \\
\therefore \mathrm{Sym}(\delta\mathbb{X}_{stu}) &= \frac{1}{2}\delta X_{st} \otimes \delta X_{tu} + \frac{1}{2}\delta X_{tu} \otimes \delta X_{st} \\
\delta(\frac{1}{2}\delta X \otimes \delta X)_{stu} &= \frac{1}{2}[\delta X_{su} \otimes \delta X_{su} - \delta X_{st} \otimes \delta X_{st} - \delta X_{tu} \otimes \delta X_{tu}] \\
&= \frac{1}{2}[\delta X_{st} \otimes \delta X_{tu} + \delta X_{tu} \otimes \delta X_{st}] \\
\therefore \delta E_{stu} &= 0
\end{aligned}$$

$\gamma \in C^{2\alpha}([0,T], \mathrm{Sym}(V \otimes V))$

"Chain Rule" Let $(Y, Y') \in \mathscr{D}^\alpha_{\mathbf{X}}$. We have defined previously, $Z_t = \int_0^t Y_s d\mathbf{X}_s$, $(Z, Z') \in \mathscr{D}^\alpha_{\mathbf{X}}$ with $Z' = Y$. Recall also,

$$\delta Z_{st} = Y_s\delta X_{st} + Y'_s\mathbb{X}_{st} + o(|t-s|^{3\alpha})$$

Let f smooth. We want an expression for $f(Z)$. We know that $(f(Z), f(Z)') \in \mathscr{D}^\alpha_{\mathbf{X}}$, $f(Z)' = Df(Z)Z' = Df(Z)Y$

An naive guess

$$df(Z_t) = Df(Z_t)dZ_t = Df(Z_t)Y_t d\mathbf{X}_t$$

is not correct, but does make sense as an integral since $Df(Z_t)Y_t$ is a controlled rough path.

$$\begin{aligned}
f(Z_t) - f(Z_s) &= Df(Z_s)\delta Z_{st} + \frac{1}{2}[D^2f(Z_s)] : [\delta Z_{st} \otimes \delta Z_{st}] + o(|t-s|^{3\alpha}) \\
&= Df(Z_s)Y_s\delta X_{st} + Df(Z_s)Y'_s\mathbb{X}_{st} + \underbrace{[Y_s^TD^2f(Z_s)Y_s] : [\delta X_{st} \otimes \delta X_{st}]}_{\text{correction from guess}} + o(|t-s|^{3\alpha}) \\
&= Df(Z_s)Y_s\delta X_{st} + Df(Z_s)Y'_s\mathbb{X}_{st} + [Y_s^TD^2f(Z_s)Y_s] : [\mathrm{Sym}(\mathbb{X}_{st}) = \delta\gamma_{st}] \\
&= Df(Z_s)Y_s\delta X_{st} + [Df(Z_s) + Y_s^TD^2f(Z_s)Y_s]\mathbb{X}_{st} - Y_s^TD^2f(Z_s)Y_s\delta\gamma_{st}
\end{aligned}$$

We can drop the Sym from $\mathrm{Sym}(\mathbb{X}_{st})$ since $Y_s^TD^2f(Z_s)Y_s$ is a symmetric matrix. If A is symmetric and B is antisymmetric, $tr(AB) = tr((AB)^T) = tr(B^TA^T) = tr(A^TB^T) = -tr(AB) = 0$.

Define $W_t = df(Z_t)Y_t$, $W'_t = Df(Z_t)Y'_t + Y_t^TD^2f(Z_t)Y_t$. This can be shown from $\delta W_t = W'_s\delta X_{st} + \underbrace{R^W_{st}}_{\lesssim|t-s|^{2\alpha}}$. Then

$$f(Z_t) - f(Z_s) = W_s\delta X_{st} + W'_s\mathbb{X}_{st} - Y_s^TD^2f(Z_s)Y_s : \delta\gamma_{st} + o(|t-s|^{3\alpha})$$

Since $Y_s^T D^2 f(Z_s) Y_s \in C^\alpha$ and $\gamma \in C^{2\alpha}$, we can define this as Young integral. Thus,

$$f(Z_t) - f(Z_s) = \int_s^t (W_r, W_r') d\mathbf{X}_r - \int_s^t Y_r^T D^2 f(Z_r) Y_r d\gamma_r$$

This is made rigorous by the sewing lemma.

Example 4.1. Consider the Brownian motion B with the Itô lift, $\mathbf{B}^I = (B, \mathbb{B}^I)$. We know that $\mathrm{Sym}(\mathbb{B}_{st}^I) = \frac{1}{2}\delta B_{st} \otimes \delta B_{st} - \frac{t-s}{2} Id \Rightarrow \gamma_t = -\frac{t}{2} Id$. Thus, for $(Y, Y') \in \mathcal{D}_{\mathbf{B}}^\alpha$ and $dZ_t = Y_t \cdot dB_t$ and f smooth, we have Itô's formula

$$df(Z_t) = Df(Z_t) Y_t dB_t + \frac{1}{2} tr(Y_t^T D^2 f(Z_t) Y_t) dt$$

For $d\tilde{Z}_t = Y_t \circ dB_t$ Stratonovich lift, $\gamma = 0$ so $df(\tilde{Z}_t) = Df(\tilde{Z}_t) Y_t dB_t$ obeys the chain rule.

4.2 Gaussian measure

Example 4.2. Let $\mathcal{B}$ be a separable Hilbert space ($\mathcal{B}^* = \mathcal{B}$).

Theorem 4.2.

Let $\mathcal{B}$ be a separable Hilbert space ($\mathcal{B}^* = \mathcal{B}$).

Definition 4.2. $T : H \to H$, a continuous, linear operator is *trace class* if $\sum_{n=1}^{\infty} \langle Te_n, e_n \rangle_H < \infty$ for some orthonormal basis (e_n)

Theorem 4.3. $\tilde{\Sigma}_\mu : B \to B$ is trace class.

Proof. Let (e_n) be an orthonormal basis of $\mathcal{B}$. Then

$$\langle \Sigma_\mu e_n, e_n \rangle = \int \langle x, e_n \rangle^2 \mu(dx) = \sum_{n=1}^{\infty} \Sigma_\mu \langle e_n, e_n \rangle = \int ||x||^2 \mu(dx) < \infty$$

by the monotone convergence theorem.

Further, given any nonnegative, symmetric, trace-class operator $T : \mathcal{B} \to \mathcal{B}$, then $\exists$ a centered Gaussian measure μ on $\mathcal{B}$ with $T = \tilde{\Sigma}_\mu$. □

In general, for $\mathcal{B}$ a separable Banach space, $\tilde{\Sigma}_\mu : \mathcal{B}^* \to \mathcal{B}$ is compact.

Example 4.3. Let $\mathcal{B} = C([0,T], \mathbb{R})$ with sup norm. Let μ be the Wiener measure. Is μ Gaussian? $\mathcal{B}^* =$ finite Borel measure on $[0,T]$. For instance, $\delta_t \in \mathcal{B}^*$ where $\delta_t(x) = x(t)$ are "evaluation measures". In general, $\ell(x) = \int_0^T x(s)\ell(ds)$ where $x(s)$ is a Brownian motion path. If we approximate this integral by step functions, it is clear that it is the sum of independent Gaussian random variables and this is true in the limit as well. Let $x \in \mathcal{B}$.

$$\Sigma_\mu(\ell, \tilde{\ell}) = \int \ell(x)\tilde{\ell}(x)\mu(dx) = \mathbb{E}\left[\int_0^T B(s)\ell(ds) \int_0^T B(s)\tilde{\ell}(ds)\right]$$
$$= \int_0^T \int_0^T \mathbb{E}[B(s)B(t)]\ell(ds)\tilde{\ell}(dt) = \int_0^T \int_0^T s \wedge t \ell(ds)\tilde{\ell}(dt)$$

Assume $\ell(ds) = f(s)ds$, Then

$$\tilde{\Sigma}_\mu(\ell)(t) = \int x\ell(x)\mu(dx)(t) = \mathbb{E}\left[B(t) \int_0^T B(s)\ell(ds)\right] = \int_0^T (s \wedge t)\ell(ds) = h(t)$$
$$h'(t) = \frac{d}{dt}\left[\int_0^t sf(s)\, ds + t \int_t^T f(s)\, ds\right] = tf(t) + \int_t^T f(s)ds - tf(t) = \int_t^T f(s)\, ds$$

[Note: finite linear combinations of δ are dense in the space of measures]

4.3 Cameron-Martin Space

Define

$$\tilde{\mathcal{H}}_\mu = \tilde{\Sigma}_\mu(\mathcal{B}^*) = \{h \in \mathcal{B} : \exists\, h^* \in \mathcal{B}^*, \tilde{\Sigma}_\mu(h^*) = h (\Leftrightarrow \Sigma_\mu(h^*, \ell) = \ell(h)\ \forall\ \ell \in \mathcal{B}^*\}$$

Given $h, \tilde{h} \in \tilde{\mathcal{H}}_\mu$, define $\langle h, \tilde{h} \rangle_{\mathcal{H}_\mu} = \Sigma_\mu(h^*, \tilde{h}^*)$ where $\Sigma_\mu h^* = h$ and $\Sigma_\mu \tilde{h}^* = \tilde{h}$. Then we can define a norm $||h||_{\mathcal{H}_\mu}^2 = \langle h, h \rangle_{\mathcal{H}_\mu} = \Sigma_\mu(h^*, h^*)$.

Suppose $\tilde{\Sigma}_\mu h^* = \tilde{\Sigma}_\mu \tilde{h}^* = h$. Then

$$\Sigma_\mu(h^*, h^*) = h^*(h) = h^*(\tilde{\Sigma}_\mu \tilde{h}^*) = \tilde{\Sigma}_\mu(\tilde{h}^*, h^*) = \tilde{h}^*(\tilde{\Sigma}_\mu h^*) = \tilde{h}^*(h) = \Sigma_\mu(\tilde{h}^*, \tilde{h}^*)$$

so the norm is well-defined.

Note: $\tilde{\mathcal{H}}_\mu$ *need not be complete.*

Let $\mathcal{H}_\mu$ be the completion of $\tilde{\mathcal{H}}_\mu$ under $||\cdot||_{\mathcal{H}_\mu}$.

Claim: $\mathcal{H}_\mu \subset \mathcal{B}$.

Proof. Let $h \in \tilde{\mathcal{H}}_\mu$. (We want control of $\mathcal{B}$ norm in terms of $\mathcal{H}_\mu$ norm)

$$||h||_{\mathcal{B}}^2 = \sup_{\ell \in \mathcal{B}^*, ||\ell||=1} \ell(h)^2 = \sup_{||\ell||=1} \Sigma_\mu(\ell, h^*) \leq \sup_{||\ell||=1} \Sigma_\mu(\ell,\ell) \underbrace{\Sigma_\mu(h^*,h^*)}_{||h||^2_{\mathcal{H}_\mu}} \leq ||\Sigma_\mu|| ||h||^2_{\mathcal{H}_\mu}$$

□

$\mu =$ centered Gaussian measure, $\Sigma_\mu : \mathcal{B}^* \to \mathcal{B}$, $\tilde{\mathcal{H}}_\mu = \Sigma_\mu(\mathcal{B}^*)$ where for $h \in \tilde{\mathcal{H}}_\mu$, $||h||^2 = \Sigma_\mu(h^*, h^*)$ where $\Sigma_\mu h^* = h$. [Note: We drop the $\sim$ to differentiate between Σ_μ as a map and as a bilinear operator. This should be clear from context.]

Remark 4.2. $h \mapsto h^*$ not uniquely defined. For example, $\mu = \delta_0$. Then $\Sigma_\mu \equiv 0$ and $\mathcal{H}_\mu = \{0\}$ so $\Sigma_\mu h^* = 0 \ \forall \ h^* \in \mathcal{B}^*$.

In general, if $\Sigma_\mu \ell = \Sigma_\mu \tilde{\ell}$ then

$$\int |(\ell - \tilde{\ell})(x)|^2 \mu(dx) = \Sigma_\mu(\ell - \tilde{\ell}, \ell - \tilde{\ell}) = (\ell - \tilde{\ell})(\Sigma_\mu(\ell - \tilde{\ell})) = 0$$

Thus, we have an isometry from $\mathcal{H}_\mu \overset{i}{\hookrightarrow} L^2(\mathcal{B}, \mu)$ where $\tilde{\mathcal{H}}_\mu \ni h \mapsto h^*$ and $||h||^2_{\mathcal{H}_\mu} = \Sigma_\mu(h^*, h^*) = ||h||_{L^2(\mu)}$.

Define the *reproducing kernel Hilbert Space* $\mathcal{R}_\mu = i(\mathcal{H}_\mu) \subset L^2(\mathcal{B}, \mu) =$ space of square integrable random variables. Recall, by letting $\mathcal{B} = \Omega$, $\mu = \mathbb{P}$, $X(\omega) = \omega$ we can make μ be the distribution of random variable.

Then $\ell \in \mathcal{R}_\mu$ $(\ell = ih) \Rightarrow \ell$ is centered Gaussian with variance $||\ell||^2_{L^2(\mu)} = ||h||^2_{\mathcal{H}_\mu}$.

Example 4.4. $\mathcal{B} = C([0,T], \mathbb{R})$, $\mu =$ Wiener measure

Exercise: If $f \in L^1([0,T]) \subset \mathcal{B}^*, \Sigma_\mu f = h \in \mathcal{B}$, then $h(t) = \int_0^t \left[\int_s^T f(r)\, dr\right] ds$. That is

$$\begin{cases} -h''(t) = f(t) \\ h(0) = 0 \\ h'(T) = 0 \end{cases} \qquad \text{and} \quad ||h||^2_{\mathcal{H}_\mu} = \int_0^T f(t)h(t)dt = \int_0^T |h'(t)|^2 dt$$

$\Rightarrow \mathcal{H}_\mu = \{h$ absolutely continuous function with $h(0) = 0$ and $h' \in L^2\}$. $\tilde{\mathcal{H}}_\mu = \{h$ absolutely continuous function with $h(0) - 0, h' \in L^2 \cap BV\}$ so $\tilde{\mathcal{H}}_\mu \subsetneq \mathcal{H}_\mu$

Proposition 4.1. If μ, ν are centered Gaussian measures on $\mathcal{B}$ and $\mathcal{H}_\mu = \mathcal{H}_\nu$ and $||h||_\mu = ||h||_\nu \ \forall \ h \in \mathcal{H}_\mu$. Then $\mu = \nu$.

Proof. Since we already know that μ and ν are centered Gaussians, it suffices to show $\ell^*\mu = \ell^*\nu \ \forall \ \ell \in \mathcal{B}^* \Rightarrow$ it suffices to show that $\int \ell(x)^2 \mu(dx) = \int \ell(x)^2 \nu(dx)$. □

Let μ be a centered Gaussian, $h \in \mathcal{B}$. Define $T_h : \mathcal{B} \to \mathcal{B}$ where $y \mapsto y + h$. Then we can consider the pushforward, $T_h^*\mu$, a measure on $\mathcal{B}$. For example, $\mathcal{B} = \mathbb{R}^d$, Σ_μ invertible $\Leftrightarrow \Sigma_\mu(\ell,\ell) > 0 \ \forall \ \ell \neq 0$.

Then take $f \in C(\mathbb{R}^d) \cap L^1(\mu)$ so

$$\begin{aligned} \int f(x)(T_h^*\mu)(dx) &= \frac{1}{(2\pi)^{d/2}\det(\Sigma)^{1/2}} \int f(x+h) e^{-\frac{1}{2}\Sigma^{-1}(x,x)}\, dx \\ &= \frac{1}{(2\pi)^{d/2}\det(\Sigma)^{1/2}} \int f(x) e^{-\frac{1}{2}\Sigma^{-1}(x-h,x-h)} dx \\ &= \frac{1}{(2\pi)^{d/2}\det(\Sigma)^{1/2}} \int f(x) e^{\Sigma^{-1}(h,x) - \frac{1}{2}\Sigma^{-1}(h,h)} \mu(dx) \end{aligned}$$

$\Rightarrow dT_h^*\mu = \exp\left(\Sigma^{-1}(h,x) - \frac{1}{2}\Sigma^{-1}(h,h)\right)\mu(dx)$ so $T_h^*\mu$ is absolutely continuous with respect to μ.

$x \in \mathcal{B}$, $h \in \mathcal{H}_\mu \to i(h) \in \mathcal{R}_\mu \subset L^2(\mathcal{B}, \mu)$ that is "$\Sigma i(h) = h$"

$$dT_h^*\mu = \exp\left(ih(x) - \frac{1}{2} \underbrace{ih(h)}_{\int i(h)^2(x)\mu(dx) = ||h||^2_{\mathcal{H}_\mu}} \right) d\mu = \exp(ih(x) - \frac{1}{2}||h||^2_{\mathcal{H}_\mu})$$

Theorem 4.4. $\dim \mathcal{H}_\mu = +\infty$. Then $T_h^*\mu$ is absolutely continuous with respect to $\mu \ \Leftrightarrow h \in \mathcal{H}_\mu$.

Proof. NTS $T_h^*\mu = \exp\left(i(h)(x) - \frac{1}{2}||h||^2_{\mathcal{H}_\mu}\right) d\mu$. We can do this by taking the Fourier transform and showing that the Fourier transforms are the same. □

Covariance is a very natural inner product.

We could have proceeded as follows:

$$\text{Fernique's thm} \Rightarrow \mathcal{B}^* \subset L^2(\mathcal{B}, \mu) \Rightarrow \mathcal{R}_\mu = \overline{\mathcal{B}^*}^{||\cdot||_{L^2}} \text{ a Hilbert space}$$

where we identify functionals in $\mathcal{B}^*$ by $\mu-$a.e. equivalence and $||\ell||^2_{L^2(\mu)} = \Sigma_\mu(\ell,\ell)$.

$\Sigma_\mu : \mathcal{B}^* \to \tilde{\mathcal{H}}_\mu \subset \mathcal{B}$ where again we take $\mathcal{B}^*$ up to $\mu-$a.e. equivalence. Σ_μ extends to an isometry $\mathcal{R}_\mu \to \mathcal{H}_\mu$ and $i : \mathcal{H}_\mu \to \mathcal{R}_\mu$ goes the other way.

Recall $L^2(\mu, \mathcal{B})$ is a collection of random variables. Hence, $\mathcal{R}_\mu$ are random variables, in fact they are Gaussian.

Example 4.5. Let $\mathcal{B} = C([0,T]), \mu =$ Wiener measure. Then $\mathcal{R}_\mu = \left\{\int_0^T f(t)dW_t : f \in L^2[0,T]\right\}$ and $\mathcal{H}_\mu = \left\{h(t) = \int_0^t f(s)ds : f \in L^2[0,T]\right\}$.

[Note: The integrals are defined since $f \in L^2$ and deterministic and therefore progressively measurable etc.]

[Note: $\mathcal{R}_\mu$ is sometimes call the *First Wiener Chaos*]
[Note: $i : \mathcal{H}_\mu \to \mathcal{R}_\mu,\ h \mapsto \int_0^T h'(s)dW_s$]

Lemma 4.1. μ, ν centered Gaussian on $\mathcal{B}$. $\mathcal{H}_\mu = \mathcal{B}, \mathcal{H}_\mu = \mathcal{H}_\nu$ and $||\cdot||_\mu = ||\cdot||_\nu$ on $\mathcal{H}_\mu$. Then $\mu = \nu$

Proof. Take $\ell \in \mathcal{B}^*$. It suffices to show $\ell^*\mu = \ell^*\nu$.

$\Sigma_\mu(\ell,\ell) = \ell(h)$ where $h = \Sigma_\mu \ell \in \mathcal{H}_\mu = ||h||^2$. Also, $h \in \mathcal{H}_\nu$. We don't know that $h \in \tilde{\mathcal{H}}_\nu$. Let $h^* = i_\nu h$. There exists $\{\ell_n\} \subset \mathcal{B}^*$ such that $||\ell_n - h^*||_{L^2(\nu)} \to 0$. Take $h_n = \Sigma_\nu \ell_n$. Then

$$\ell(h) = \ell(h_n) + \ell(h - h_n) \leq \Sigma_\nu(\ell, \ell_n) + ||\ell||_{\mathcal{B}^*}||h - h_n||_{\mathcal{B}}$$

Since convergence in $\mathcal{H}_\nu$ implies convergence in $\mathcal{B}$.

$$\Sigma_\nu(\ell,\ell_n) \leq ||\ell||_{L^2(\nu)}||\ell_n||_{L^2(\nu)} \to \underbrace{||\ell||^2_{L^2(\mu)}}_{||h||^2_{\mathcal{H}_\nu}} = \Sigma_\mu(\ell,\ell) \leq ||\ell||_{L^2(\nu)} \underbrace{||h^*||_{L^2(\nu)}}_{||h||_{\mathcal{H}_\nu} = ||h||_{\mathcal{H}_\mu}}$$

So, $||\ell||_{L^2(\mu)} \leq ||\ell||_{L^2(\nu)}$ and similarly, $||\ell||_{L^2(\nu)} \leq ||\ell||_{L^2(\mu)}$

Thus, $\ell^*\mu$ and $\ell^*\nu$ are both centered Gaussian measures with the same variance, $||\ell||_{L^2(\mu)} = ||\ell||_{L^2(\nu)}$, so $\ell^*\mu = \ell^*\nu$. ☐

Lemma 4.2. $h \in \mathcal{H}_\mu$, $\ell \in \mathcal{B}^*$. Then $\ell(h) = \langle \ell, ih\rangle_{L^2(\mu)}$

Remark 4.3. If $h \in \tilde{\mathcal{H}_\mu}$, this is essentially by definition. In other words, ih need not be in $\mathcal{B}^*$ for "$\ell(h) = \Sigma_\mu(\ell, ih)$"

Proof. Let $h_n \to h$ in $\mathcal{H}_\mu$ where $h_n \in \tilde{\mathcal{H}}_\mu$. Then

$$\ell(h_n) = \underbrace{\Sigma_\mu(\ell, ih_n)}_{\langle \ell, ih\rangle_{L^2(\mu)}} \Rightarrow \ell(h) = \langle \ell, ih\rangle_{L^2(\mu)}$$

☐

Lemma 4.3. For $x \in \mathcal{B}$, define $||x|| = \sup\{\ell(x) : \ell \in \mathcal{B}^*, \Sigma_\mu(\ell,\ell) \leq 1\}$. Then

1. If $x \in \mathcal{H}_\mu$, then $||x|| = ||x||_{\mathcal{H}_\mu}$. In particular, $||x|| < \infty$.
2. If $||x|| < \infty$, then $x \in \mathcal{H}_\mu$
3. $\Rightarrow \mathcal{H}_\mu = \{x \in \mathcal{B} : ||x|| < \infty\}$

Proof. 1. Let $h \in \mathcal{H}_\mu$. Let $\ell \in \mathcal{B}$, $\Sigma_\mu(\ell,\ell) \leq 1$. Then $\ell(h) = \langle \ell, ih\rangle_\mu \leq \underbrace{||\ell||_{L^2(\mu)}}_{\Sigma_\mu(\ell,\ell)}\underbrace{||ih||_{L^2(\mu)}}_{||h||_{\mathcal{H}_\mu}} \leq ||h||_{\mathcal{H}_\mu}$.

Taking the sup over all such ℓ, we get that $||h|| \leq ||h||_{\mathcal{H}_\mu} < \infty$

Take $\ell_n \overset{L^2(\mu)}{\to} ih,\ \ell_n \in \mathcal{B}^*$. Then,

$$||h|| \geq \frac{\ell_n(h)}{\Sigma_\mu(\ell_n,\ell_n)^{1/2}} = \frac{\langle \ell_n, ih\rangle_\mu}{||\ell_n||_{L^2(\mu)}} \overset{n\to\infty}{\to} \frac{||ih||_{L^2(\mu)}}{||ih||_{L^2(\mu)}}$$

[Note: If $h \in \tilde{\mathcal{H}}_\mu$, then ih maximizes the sup in $||h||$. The norm $||x||$ generalizes to when $h \notin \tilde{\mathcal{H}}_\mu$ via approximation.]

2. Assume $||x|| < \infty$. Define $X : \mathcal{B}^* \to \mathbb{R}$ where $\ell \mapsto \ell(x)$. (Copying proof that $\mathcal{B} \hookrightarrow \mathcal{B}^{**}$.

Claim: If $\ell = 0$ $\mu-$a.e. then $X(\ell) = 0$ which by linearity implies $X(\ell) = X(\tilde{\ell})$ if $\ell = \tilde{\ell}$ a.e.

ILLEGAL: $\left|X\left(\frac{\ell}{(\Sigma_\mu(\ell,\ell))^{1/2}}\right)\right| = \left|\frac{\ell(x)}{(\Sigma_\mu(\ell,\ell))^{1/2}}\right| \leq ||x|| < \infty$ so $|X(\ell)| \leq ||x||||\ell||_{L^2(\mu)}$

What if $\Sigma_\mu(\ell,\ell) = 0$? Instead take $\bar{\ell} \neq 0$ a.e. and compute,

$$\frac{\ell + \epsilon\bar{\ell}}{||\ell + \epsilon\bar{\ell}||_{L^2(\mu)}}(x)$$

Thus, X extends to a bounded linear functional on $\mathcal{R}_\mu$. So, there exists an $x^* \in \mathcal{R}_\mu : x(\ell) = \langle \ell, x^*\rangle_{L^2(\mu)}\ \forall\ \ell \in \mathcal{R}_\mu$.

Let $h = \Sigma_\mu x^*$. By definition,$\forall\ \ell \in \mathcal{B}^*$, $x(\ell) = \langle \ell, x^*\rangle_\mu = \ell(h)$ but $x(\ell) = \ell(x)$, so $x = h \in \mathcal{H}_\mu$.

☐

Corollary 4.2. If $x \notin \mathcal{H}_\mu$, then $||x|| = \infty$ i.e. there exists $\{\ell_n\} \subset \mathcal{B}^* : \Sigma_\mu(\ell_n,\ell_n) = 1$ and $\ell_n(x) \geq n$.

Theorem 4.5. (Cameron-Martin) $h \in \mathcal{B}$. Then μ and T_h^* are either

1. Absolutely continuous with respect to each other, if $h \in \mathcal{H}_\mu$.

2. Mutually singular with respect to each other if $h \notin \mathcal{H}_\mu$

Remark 4.4. 1. Suppose $\mu = \overline{\mathcal{H}_\mu}^{\mathcal{B}}$. If dim $\mathcal{H}_\mu = \infty$, then $\mu(\mathcal{H}_\mu) = 0$

2. If $\mathcal{B} = \mathbb{R}^d$, then $\mathcal{H}_\mu$ is the support of the measure.

3. Think of the Cameron Martin theorem in terms of both degenerate and nondegenerate Gaussian measures in $\mathbb{R}^d$.

Proof of part 2 of Cameron-Martin theorem (the proof of part 1 is done previously):

Proof. Let $h \notin \mathcal{H}_\mu \Rightarrow \exists (\ell_n) \subset \mathcal{B}^*$ such that $\Sigma_\mu(\ell_n, \ell_n) = 1$ and $\ell_n(h) \leq -n$. (We can do this since if $\ell_n(h) \geq n$, $-\ell_n(h) \in \mathcal{B}^*$ with the same norm.) Then it suffices to show $||T_h^* \mu - \mu||_{TV} = 2$ that is, there is no overlapp or cancellation.

By the triangle inequality, $||T_h^*\mu - \mu||_{TV} \leq 2$. Also,

$$\underbrace{||T_h^*\mu - \mu||_{TV(\mathcal{B})}}_{\text{sup over all partitions}} \geq \underbrace{||\ell_n^* T_h^* \mu - \ell_n^* \mu||_{TV(\mathbb{R})}}_{\text{only partitions over sets Borel meas images of } \ell_n}$$

Set $m_n = -\ell_n(h)$. Then

$$\begin{aligned}
||\ell_n^* T_h^* \mu - \ell_n^* \mu||_{TV(\mathbb{R})}||_{TV(\mathbb{R})} &= ||\mathcal{N}(m_n, 1) - \mathcal{N}(0,1)||_{TV(\mathbb{R})} \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left| e^{-y^2/2} - e^{(y-m_n)^2/2} \right|^2 dy \\
&= \frac{2}{\sqrt{2\pi}} \int_{-\infty}^{m_n/2} \left(e^{-y^2/2} - e^{(y-m_n)^2/2} \right)^2 dy \quad \text{(by symmetry)} \\
&= 2\mathbb{P}[\mathcal{N}(0,1) \leq m_n/2] - 2 \underbrace{\mathbb{P}[\mathcal{N}(m_n, 1) \leq m_n/2]}_{\mathbb{P}[\mathcal{N}(0,1) > m_n/2]} \\
&= 2\left(1 - 2\mathbb{P}[\mathcal{N}(0,1) > m_n/2]\right) \\
&\geq 2(1 - 2\mathbb{P}[\mathcal{N}(0,1) > n/2]) \overset{n\to\infty}{\longrightarrow} 2
\end{aligned}$$

□

[Note: The Cameron-Martin theorem tells us which direction we can go and have an equivalent measure.]

Corollary 4.3. (a) $\mathcal{H}_\mu = \bigcap \{V : V \subset \mathcal{B} \text{ a linear subspace of } \mathcal{B},\ \mu(V) = 1\}$

(b) $\mu(\mathcal{H}_\mu) = 0$ if $\dim(\mathcal{H}_\mu) = \infty$.

Proof. (a) Let $V \subset \mathcal{B}$ be a linear subspace with $\mu(V) = 1$. Let $h \in \mathcal{H}_\mu$. By the Cameron-Martin theorem, $\mu(V + h) = T_h^* \mu(V) = 1$ (that is because μ and $T_h^*\mu$ are absolutely continuous with respect to each other). Then $V \cap (V + h) \neq \emptyset \Rightarrow h \in V \Rightarrow \mathcal{H}_\mu \subset V$.

Now, $x \notin \mathcal{H}_\mu \Rightarrow \exists (\ell_n) \subset \mathcal{B}^*$, $||\ell_n||_{L^2(\mu)} = 1$ and $\ell_n(x) \geq n$. Define $V = \left\{ y \in \mathcal{B} : |y|^2 = \sum_{n=1}^{\infty} \frac{|\ell_n(y)|^2}{n^2} < \infty \right\}$.

Then $|x|^2 = \sum_{n=1}^{\infty} \frac{|\ell_n(x)|^2}{n^2} \geq \sum_{n=1}^{\infty} 1 = +\infty$ so $x \notin V$. However, $\int |y|^2 \mu(dy) = \sum_{n=1}^{\infty} \frac{1}{n^2} \underbrace{\int |\ell_n(y)|^2 \mu(dy)}_{1} < \infty$

so $|y| < \infty$ $\mu-$a.e. $\Rightarrow \mu(V) = 1$.

(b) Let $(e_n^*) \subset \mathcal{R}_\mu (\subset L^2(\mathcal{B}, \mu))$ be an orthonormal basis. We know that (e_n) are $\mathcal{N}(0,1)$ and orthogonal, $\mathbb{E}[e_n^*(\omega) e_m^*(\omega)] = \delta_{mn}$. Further, we know for $n \neq m$, (e_n^*, e_m^*) is a Gaussian vector with covariance matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so (e_n^*) are independent.

[Note: Orthogonality does not imply independence in general, this is only due to (e_n^*) being Gaussian.]

We also know $\limsup_{n\to\infty} e_n^*(x) = +\infty$. (If I wait long enough, it will be large: Borel Cantelli)

Since $ie_n = e_n^*$, (e_n) is an orthonormal basis of $\mathcal{H}_\mu$.

Yet, $x \in \mathcal{H}_\mu$, $||x||^2_{\mathcal{H}_\mu} = \sum_{n=1}^{\infty} \langle x, e_n \rangle^2_{\mathcal{H}_\mu} = \sum_{n=1}^{\infty} e_n^*(x)^2 < \infty$.

For these two things to be true simultaneously, it must be that $\mu(\mathcal{H}_\mu) = 0$. The complement of where $\limsup e_n^*(x) = +\infty$ has measure 0 and we have show that $\mathcal{H}_\mu$ is a subset of this set.

□

Recall: If $x \in \mathcal{H}_\mu$ and $\ell = 0$ $\mu-$a.e., then $\ell(x) = 0$. This is kind of crazy as it says that the information contained in a null set dictates almost everything. This is because of the structure of the measure μ.

Theorem 4.6. Let (e_n) be an orthonormal basis for $\mathcal{H}_\mu$. Let $(\xi_n)_{n=1}^{\infty}$ i.i.d. $\mathcal{N}(0,1)$. Define $X_N(\omega) = \sum_{n=1}^{N} \xi_n(\omega) e_n$ (X_N is a $\mathcal{B}-$valued random variable). Then with probability 1, $X_N \overset{N\to\infty}{\longrightarrow} X$ in $\mathcal{B}$ where X is a $\mathcal{B}-$valued random variable with law μ.

[Note: With probability 1, X does not take values in $\mathcal{H}_\mu$, but in the bigger space $\mathcal{B}$.]

Proof. $\ell \in \mathcal{B}$, $\ell(X_N(\omega)) \to \ell(X)$ a.s. Take $(\Omega, \mathbb{P}) = (\mathcal{B}, \mu)$ and $\xi_n = e_n^* = ie_n$. Then $X_N(x) = \sum_{n=1}^{N} e_n^*(x)e_n \to X(x) = x$.

Look at $\ell(X_N(x)) = \sum_{n=1}^{N} e_n^*(x)\ell(e_n) = \sum_{n=1}^{N} \langle \ell, e_n^* \rangle_{L^2(\mu)} e_n^*(x) \overset{N\to\infty}{\longrightarrow} \ell(x)$ in $L^2(\mu)$. ☐

Lemma 4.4. μ a Gaussian measure on a separable Banach space $\mathcal{B}$. Let $\mathcal{B}' \hookrightarrow \mathcal{B}$ (continuous inclusion) be a separable Banach space such that $\mu(\mathcal{B}') = 1$. Then, $\mathcal{H}_{\mu|_{\mathcal{B}'}} = \mathcal{H}_\mu$

Proof. (sketch) $L^2(\mathcal{B}, \mu) = L^2(\mathcal{B}', \mu|_{\mathcal{B}'}$ but it is not clear that $\overline{\mathcal{B}^*} = \overline{(\mathcal{B}')^*}$ only that $\overline{\mathcal{B}^*} \subset \overline{(\mathcal{B}')^*}$.

$$\mathcal{H}_\mu = \bigcap \{V \in \mathcal{B} : \mu(V) = 1\} \text{ and } \mathcal{H}_{\mu|_{\mathcal{B}'}} = \bigcap \{V \subset \mathcal{B}' : \mu(V) = 1\} \supset \mathcal{H}_\mu$$

However $\mathcal{B}'$ is such that $\mu(\mathcal{B}') = 1$, so $\mathcal{H}_\mu = \bigcap \{V \cap \mathcal{B}' : \mu(V) = 1\} = \mathcal{H}_{\mu|_{\mathcal{B}'}}$

Now, we need to show $||h||_{\mathcal{H}_\mu} = ||h||_{\mathcal{H}_{\mu|_{\mathcal{B}'}}}$.

$h \in \mathcal{H}_\mu \Rightarrow h = \int_{\mathcal{B}} x\ell(x)\mu(dx) = \int_{\mathcal{B}'} x\ell(x)\mu(dx)$ where $\ell \in \mathcal{R}_\mu = \overline{\mathcal{B}^*}^{L^2}$ is unique ($\ell = ih$). $ih = \ell \in \overline{\mathcal{B}^*} \subset \overline{(\mathcal{B}')^*} = \mathcal{R}_{\mu|_{\mathcal{B}'}}$ and $||h||_{\mathcal{H}_\mu} = ||\ell||_{L^2(\mu)}$ (L^2 in $\mathcal{B}$ and $\mathcal{B}'$), so $||h||_{\mathcal{H}_{\mu|_{\mathcal{B}'}}}$. ☐

Thus, the $\mathcal{B}$ on which we define μ doesn't matter as long as it contains the support of μ.

Theorem 4.7. (Lyons)(Motivation for Rough Path theory) $\exists!$ separable Banach space $\mathcal{B}$ such that $C^1([0,1], \mathbb{R}) \hookrightarrow \mathcal{B} \hookrightarrow C^0([0,1], \mathbb{R})$, the Wiener measure μ is supported in $\mathcal{B}$, and $(C^1 \times C^1) \ni (f, g) \mapsto \int_0^1 f(g)g'(s)ds$ extends continuously to $\mathcal{B} \times \mathcal{B}$.

[Note: $\{\text{BM paths}\} \overset{a.s.}{\subset} C_0^\alpha = \overline{C^1}^{||\cdot||_{C^\alpha}}$ is a separable Banach space but does not satisfy the last hypothesis of the theorem.]

Proof. $(\mu, \mathcal{B}) \to \mathcal{H}_\mu = \{h : h(0) = 0, h' \in L^2([0,1])\} \subset H^1([0,1])$ and $||h||^2_{\mathcal{H}_\mu} = \int_0^1 |h'_s|ds$. (This is a norm since $h(0) = 0$ and the P.C. inequality.

Define $e_0(t) = t$ and $e_n(t) = \frac{\sin(2\pi nt)}{\sqrt{2}\pi n}$, $e_{-n}(t) = \frac{1-\cos(2\pi nt)}{\sqrt{2}\pi n}$ for $n \in \mathbb{N}$ (Fourier Modes). $[e_n(0) = 0, e_n(1) = 0 \ \forall\ n \neq 0]$ Then $\{e_n\}$ is ONB for $\mathcal{H}_\mu$. Let $\{\xi_n\}$ be iid $\mathcal{N}(0,1)$. Define $X_N(t) = \sum_{0<|n|\leq N} \xi_n e_n(t)$ and $Y_N(t) = \sum_{0<|n|\leq N} -\underbrace{sgn(n)\xi_{-n}}_{\tilde{\xi_n}} e_n(t)$.

Then $\mu-$a.s. $X_N \to X$ and $Y_N \to Y$ in $\mathcal{B}$. [Note: X and Y are actually brownian bridges since we left out the $n = 0$ terms]

$\xi_0 t + X$ and $\xi_0 t + Y$ are BM. (uses that $\mu(\mathcal{B}) = 1$)

Assume that we can integrated on $\mathcal{B} \times \mathcal{B}$. Then $\mu - a.s.$

$$I = \int_0^1 X_N(t)Y'_N(t)dt \to \text{something finite}$$

For $n > 0, e'_n(t) = \sqrt{2}\cos(2\pi nt), e'_{-n}(t) = \sqrt{2}\sin(2\pi nt)$

$$\begin{aligned} I &= \sum_{n=1}^{N} \xi_n^2 \int_0^1 \frac{\sin^2(2\pi nt)}{\pi n} dt + \sum_{n=1}^{N} \xi_{-n}^2 \int_0^1 \frac{\cos^2(2\pi nt)}{\pi n} dt \\ &= \sum_{n=1}^{N} \frac{\xi_n^2}{2\pi n} + \sum_{n=1}^{N} \frac{\xi_{-n}^2}{2\pi n} \\ &= \sum_{n=1}^{N} \frac{\xi_n^2 + \xi_{-n}^2}{2\pi n} \end{aligned}$$

(NTS that $\mu - a.s.$ this diverges).

$$\mathbb{P}\left[\sum_{n=1}^{N} \frac{\xi_n^2}{2} \leq (1-\epsilon) \sum_{n=1}^{N} \frac{1}{n}\right] = \mathbb{P}\left[\sum_{n=1}^{N} \frac{\xi_n^2 - 2}{n} \leq -\epsilon L_N\right]$$

where $L_N = \sum_{n=1}^{N} \frac{1}{n}$. Then, by Chebyshev,

$$\mathbb{P}\left[\sum_{n=1}^{N} \frac{\xi_n^2 - 2}{n} \leq -\epsilon L_N\right] \leq \frac{\mathbb{E}\left|\sum_{n=1}^{N} \frac{\xi_n^2 - 1}{n}\right|^2}{\epsilon^2 L_N}$$

Since $\mathbb{E}[\xi_n^2] = 1$, $\mathbb{E}[\xi_n^2 - 1] = 0$. By independent, $\sum_{n=1}^{N} \mathbb{E}[|\xi_n^2 - 1|] = C$, so

$$\frac{\mathbb{E}\left|\sum_{n=1}^{N} \frac{\xi_n^2 - 1}{n}\right|^2}{\epsilon^2 L_N} \leq \frac{C}{\epsilon^2(\log(N))^2}$$

so the sequences goes to ∞ in measure.

We can pick a subsequence such that the terms are summable and apply Borel-Cantelli to show that the subsequence converges a.s. Then use monotonicity of the sum to show that the sequence converges a.s. ☐

White noise: Random "function" $\xi : \Omega \times \Pi^d \to \mathbb{R}$ such that $\mathbb{E}[\xi(x)\xi(y)] = \delta_0(x-y)$. As a distribution $\mathbb{E}[\langle \xi, \varphi\rangle \langle \xi, \psi\rangle] = \langle \varphi, \psi\rangle_{L^2}$. Perhaps $\xi \in L^2(\Pi^d)$?

Let $\{e_n\}$ be a Fourier basis, $\xi(x) = \sum_{n\in\mathbb{Z}^d} \xi_n e_n(x)$ for some ξ_n iid $\mathcal{N}(0,1)$. then μ a centered Gaussian measure on $L^2(\Pi^d) = \mathcal{B} = \mathcal{B}^*$ where $\mu(A) = \mathbb{P}(\xi(\cdot) \in A)$.

Then $\Sigma_\mu(\varphi,\psi) = \langle \varphi, \psi\rangle_{L^2}$. Then $\Sigma_\mu \varphi = \varphi$ since

$$\langle \Sigma_\mu \varphi, \psi\rangle - \Sigma_\mu(\varphi,\psi) = \langle \varphi,\psi\rangle.$$

By Id is not compact, so ξ cannot be $L^2(\Pi^d)-$ valued.

What we'll do is take $\mathcal{H}_\mu = L^2(\Pi^d)$ and find a suitable $\mathcal{B}$ with $\mathcal{H}_\mu$ as is Cameron-Martin space.

We want to find a space where the Cameron-Martin space is L^2. Define

$$H^{-s}(\Pi^d) = \left\{ f \in \mathcal{D}'(\Pi^d) : \underbrace{\sum_{k\in\mathbb{Z}^d} \frac{|\hat{f}(x)|^2}{(1+|k|^2)^s}}_{||f||^2_{H^{-s}}} < \infty \right\}$$

Then $\xi(x) = \sum_{k\in\mathbb{Z}^d} \xi_k e_k(x)$ where (e_k) is Fourier basis of $L^2(\Pi^d)$ is in H^{-s} for $s > d/2$ (since $\mathbb{E}[||\xi||^2_{H^{-s}}] = \sum_{k\in\mathbb{Z}^d} \frac{1}{(1+|k|^2)^s}] < \infty$ for $s > d/2$).

For $\varphi \in H^{-s}$, $\frac{\hat{\varphi}(k)}{(1+|k|^2)^s} = \widehat{(Id-\Delta)^{-s}\varphi}(k)$, "undoing 2 derivatives". So for $\varphi,\psi \in H^{-s}$,

$$\langle \varphi,\psi\rangle_{H^{-s}} = \langle (Id-\Delta)^{-s}\varphi, \psi\rangle_{L^2}.$$

So, we define $\mu(A) = \mathbb{P}(\xi \in A)$ for $A \subset H^{-s} \Rightarrow \mu$ is Gaussian measure on $H^{-s}(\Pi^d)$. Then the covariance operator, $\varphi,\psi \in (H^{-s})^* = H^{-s}$,

$$\Sigma_\mu(\varphi,\psi) = \mathbb{E}[\langle \varphi, \xi\rangle_{H^{-s}} \langle \psi, \xi\rangle_{H^{-s}}] = \sum_{k\in\mathbb{Z}^d} \frac{\hat{\varphi}(k)\hat{\psi}(k)}{(1+|k|^2)^{2s}} = \langle (Id-\Delta)^{-s}\varphi, \psi\rangle_{H^{-s}}$$

since the cross terms vanish (independence). Thus, $\Sigma_\mu = (Id-\Delta)^{-s} : H^{-s} \to H^{-s}$.

$L^2(\Pi^d) \overset{i}{\hookrightarrow} H^{-s}(\Pi^d)$, then we also have the adjoint map, $(H^{-s}(\Pi^d))^* = H^{-s} \overset{i^*}{\to} (L^2(\Pi^d))^* = L^2(\Pi^d)$. So, for $\varphi \in L^2$ and $f \in H^{-s}$

$$\langle i^* f, \varphi\rangle_{L^2} = \langle f, i\varphi\rangle_{H^{-s}} = \langle (Id-\Delta)^{-s} f, \varphi\rangle_{L^2} \Rightarrow i^* = (Id-\Delta)^{-s} \Rightarrow \Sigma_\mu = ii^*$$

The Cameron-Martin Space is L^2: starting with $\varphi \in \tilde{\mathcal{H}}_\mu \Rightarrow \exists f \in H^{-s}$ such that $\varphi = (Id-\Delta)^{-s} f \in H^s$ (add $2s$ derivatives) and norm

$$||\varphi||_{H_\mu} = \Sigma_\mu(f,f) = \langle (Id-\Delta)^{-s} f, f\rangle_{H^{-s}} = \langle (Id-\Delta)^{-s} f, (Id-\Delta)^{-s} f\rangle = ||\varphi||^2_{L^2}$$

Then $H_\mu = \overline{H^s}^{L^2} = L^2$.

- *White noise over a general Hilbert space, H*: (1) find Hilbert space K such that $H \overset{i}{\hookrightarrow} K$ where $ii^* : K \to K$ is trace class, (2) define μ to have covariance operator ii^*, (3) then $\sum_n \xi_n e_n \sim \mu$ where (e_n) is ONB of H.
- *Spacetime white noise on* $[0,T]\times\Pi^d$ "$W(t,x) = \int_0^t \xi(s,x)ds$" where "$\xi(t,x)$". Then $\mathbb{E}[W(s,x)W(t,y)] = (s\wedge t)\delta(x-y) \Rightarrow W(t,x) = \sum_{k\in\mathbb{Z}^d} W_t^k e_k(x)$ where $(W^k)_{k\in\mathbb{Z}^d}$ independent Brownian motions. W is called a cylindrical Brownian motion over $L^2(\Pi^d)$ which is specified by a Gaussian measure μ on $C([0,T], H^{-s}(\Pi^d)), s > d/2$. We only need the covariance to specify this measure.

  $\varphi,\psi \in H^{-s}$, $\mathbb{E}[\langle W_s, \varphi\rangle_{H^{-s}} \langle W_t, \psi\rangle_{H^{-s}}] = (s\wedge t)\langle (Id-\Delta)^{-s}\varphi, \psi\rangle_{H^{-s}}$

  $\mathcal{H}_\mu = H_0^1([0,T], L^2(\Pi^d))$ (where here the 0 indicates that $h(t=0) = 0$.

Example 4.6. (Stochastic Heat Equation)

$$du = (u_{xx} - \lambda u)dt + \sigma dW$$

where W is cylindrical BM over $L^2(\Pi^d)$. We can completely diagonalize this operator, $u(t,x) = \sum_{k\in\mathbb{Z}} Y_t^k e_k(x)$. For (e_k) eigenfunctions, $(\partial_x^2 - \lambda)(e_k = -(\underbrace{k^2+\lambda}_{\mu_k})e_k$. Then we get a system of SDEs, $dY_t^k = -\mu_k Y_t^k dt + \sigma dW_t$. Further, $\exists \gamma_0$ such that $Y_0^k = \gamma_0^k \sim \mathcal{N}(0,\gamma^2)$ such that $Y_0^k \sim Y_t^k$. In this case, $Y_t^k \sim \mathcal{N}\left(0, e^{-2\mu_k t}\gamma^2 + (1-e^{-2\mu_k t}\frac{\sigma^2}{2\mu_k}\right)$ so we choose $\gamma^2 = \frac{\sigma^2}{2\mu_k}$.

$|Y_k| \sim \frac{\sigma}{\sqrt{\mu_j}} \sim \frac{1}{k} \therefore u(t,\cdot) \in H^s \;\forall\; s < \frac{1}{2}$.

We can lift $u(t,\cdot)$ to Rough Path space, $(u(t,\cdot), \approxeq(t,\cdot)) \in \mathscr{C}_g^\alpha$ for $\alpha < 1/2$ (just like BM)

These notes were taken from a class taught by Ben Seeger in Spring 2024 at UT Austin.