January 2023

1.

2.

Let X be a random variable with $X \geq 0$ a.s., and suppose $\mathbb{E}[X] \leq 1$ and $\mathbb{E}[X^2] \leq 10$. Given this information, for every $t \geq 0$ find the best possible upper bound for $\mathbb{P}[X > t]$.

Solution.

$$\mathbb{E}[X] = \mathbb{E}[X\mathbf{1}_{X\leq t}] + \mathbb{E}[X\mathbf{1}_{X>t}] \geq \mathbb{E}[X\mathbf{1}_{X\leq t} + t\mathbb{P}[X > t]$$
$$\Rightarrow \mathbb{P}[X > t] \leq \frac{\mathbb{E}[X]}{t} \leq \frac{1}{t}$$
$$\mathbb{E}[X^2] = \mathbb{E}[X^2\mathbf{1}_{X\leq t}] + \mathbb{E}[X^2\mathbf{1}_{X>t}] \geq \mathbb{E}[X\mathbf{1}_{X\leq t}] + t^2\mathbb{P}[X > t]$$
$$\Rightarrow \mathbb{P}[X > t] \leq \frac{\mathbb{E}[X^2]}{t^2} \leq \frac{10}{t^2}$$

For $0 \leq t \leq 10$, $\frac{1}{t} \leq \frac{10}{t^2}$ and for $t > 10$, $\frac{10}{t^2} < \frac{1}{t}$, so

$$\mathbb{P}[X > t] \leq f(t) = \begin{cases} \frac{1}{t} & 0 \leq t \leq 10 \\ \frac{10}{t^2} & t > 10 \end{cases}.$$

3.

Let $\xi_1, \xi_2, \ldots$ be independent coin flips and define $S_n \sum_{i=1}^{n} \xi_i$.

(a) Compute $\mathbb{E}[S_{10}|\xi_1]$

(b) Compute $\mathbb{E}[S_{10}^2|\xi_1]$

(c) Compute $\mathbb{E}[\xi|S_{10}]$

Solution.

(a) $\mathbb{E}[S_{10}|\xi_1] = \sum_{i=1}^{10} \mathbb{E}[\xi_i|\xi_1] = \xi_1$

(b) $\mathbb{E}[S_{10}^2|\xi_1] = \mathbb{E}[\sum_{i=1}^{10} \xi_i^2 + 2\sum_{i\neq j} \xi_i\xi_j|\xi_1] = 10$

(c) Since ξ_i are iid, $\mathbb{E}[\xi_i|\S_{10}] = \mathbb{E}[\xi_j|S_{10}]$ for $1 \leq i, j \leq 10$. Thus,

$$\mathbb{E}[S_{10}|S_{10}] = \sum_{i=1}^{10} \mathbb{E}[\xi_i|S_{10}] = \sum_{i=1}^{10} \mathbb{E}[\xi_1|\S_{10} = 10\mathbb{E}[\xi_1|\S_{10}] = S_{10} \Rightarrow \mathbb{E}[\xi_1|S_{10}] = \frac{1}{10}S_{10}$$

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August 2022

1.

1. Show for any random variable X, and any $s, t \geq 0$,
$$\mathbb{P}[X \geq t] \leq e^{-st}\mathbb{E}[e^{sX}]$$
2. Let $\xi_1, \ldots, \xi_n$ be independent coin flips and $X_n = \sum_{i=1}^{n} \xi_n$. Prove that for any $t \geq 0$,
$$\mathbb{P}[X_n \geq t\sqrt{n}] \leq e^{-t^2/2}.$$

Solution.

1. $$\mathbb{E}[e^{sX}] = \mathbb{E}[e^{sX}\mathbf{1}_{X<t}] + \mathbb{E}[e^{sX}\mathbf{1}_{X\geq t}] \geq e^{st}\mathbb{P}[X \geq t].$$
Since $e^{sX} \geq 0$, This implies $p[X \geq t] \leq e^{-st}\mathbb{E}[e^{sX}]$.
2. By (1), $\mathbb{P}[\frac{X_n}{\sqrt{n}} \leq t] \leq e^{-t^2}\mathbb{E}[e^{tX_n/\sqrt{n}}]$, so we want to evaluate this expectation.
$$\mathbb{E}[e^{tX_n/\sqrt{n}}] = \prod_{i=1}^{n} \mathbb{E}[e^{\frac{t}{\sqrt{n}}\xi_i}] = \prod_{i=1}^{n} \frac{1}{2}\left(e^{-\frac{t}{\sqrt{n}}} + e^{\frac{t}{\sqrt{n}}}\right) = \prod_{i=1}^{n} \cosh(\frac{t}{\sqrt{n}})$$
$$= (\cosh(\frac{t}{\sqrt{n}})^n \leq ((e^{\frac{1}{2}\left(\frac{t}{\sqrt{n}}\right)^2})^n = e^{\frac{1}{2}t^2}.$$
Thus, $\mathbb{P}[X_n \geq t\sqrt{n}] \leq e^{-t^2}e^{t^2/2} = e^{-t^2/2}$.

2.

For random variables X and Y defined

$$d(X,Y) = \inf\{\epsilon \geq 0 : \mathbb{P}[|X-Y| > \epsilon] \leq \epsilon.$$

Prove that d metrizes convergence in probability, in the sense that $X_n \to X$ in probability if and only if $d(X_n, X) \to 0$.

3.

Let $\xi_1, \xi_2, \ldots$ be iid coin flips. Let $X_n = \sum_{i=1}^{n} \xi_n$, and let

$$T = \inf\{n \geq 4 : \xi_n = -1 \text{ and } \xi_{n-1} = \xi_{n-3} = 1\}.$$

1. Compute $\mathbb{E}[X_T]$
2. Compute $\mathbb{E}[X_{T+1}]$
3. Compute $\mathbb{E}[X_{T-1}]$

Solution.

1. $\mathbb{E}[X_T] = \sum_{i=1}^{T-4} \mathbb{E}[\xi_i] + \mathbb{E}[\xi_{T-3} + \xi_{T-2} + \xi_{T-1} + \xi_T] = \frac{1}{2}(0) + \frac{1}{2}(2) = 1$
2. $\mathbb{E}[X_{T+1}] = \mathbb{E}[X_T] + \mathbb{E}[\xi_{t+1}] = 1$
3. $\mathbb{E}[X_{T-1}] = \sum_{i=1}^{T-4} \mathbb{E}[\xi_i] = \mathbb{E}[\xi_{T-3} + \xi_{T-2} + \xi_{T-1}] = \frac{1}{2}(1) + \frac{1}{2}(3) = 2$

2.

Let μ be a probability measure on $\mathbb{R}$ and let φ is characteristic function. Show that μ has no atoms if

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-ita}\varphi(t)\ dt = 0 \text{ for all } a \in \mathbb{R}.$$

Solution. Let $a \in \mathbb{R}$.

$$\begin{aligned}
\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-ita}\varphi(t)\ dt &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-ita} \int_{\mathbb{R}} e^{itx}\ \mu(dx)dt = \lim_{T\to\infty} \frac{1}{2T} \int_{\mathbb{R}} \int_{-T}^{T} e^{iT(x-a}\ dt\mu(dx) \\
&= \lim_{T\to\infty} \int_{\mathbb{R}} \frac{\left(e^{iT(x-a)} - e^{-iT(x-a)}\right)}{2iT(x-a)} \mu(dx) \\
&= \lim_{T\to\infty} \int_{\mathbb{R}} \frac{\sin(T(x-a))}{T(x-a)}\ \mu(dx) \\
&= \lim_{T\to\infty} \int_{\mathbb{R}\setminus\{a\}} \frac{\sin(T(x-a))}{T(x-a)}\ \mu(dx) + \lim_{T\to\infty} \int_{\{a\}} \frac{\sin(T(x-a))}{T(x-a)}\ \mu(dx) \\
&= \int_{\mathbb{R}} \lim_{T\to\infty} \frac{\sin(T(x-a))}{T(x-a)}\ \mu(dx) = \mu(\{a\}) = 0
\end{aligned}$$

Since $a \in \mathbb{R}$ was chosen arbitarily, μ has no atoms.

January 2022

1.

Suppose that $\{X_n, n \geq 1\}$ is a sequence of iid nonnegative random variables. If $\mathbb{E}[X_1] = \infty$, show tht $\frac{1}{n} \sum_{k=1}^{n} X_k \to \infty$.

Solution. For the sake of contradiction, suppose $\frac{1}{n} \sum_{k=1}^{n} X_k \not\to \infty$. Thus, since $X_n \geq 0$ a.e., there exists a $C \geq 0$ such that $\frac{1}{n} \sum_{k=1}^{n} X_k < C\ \forall\ n$. Thus, $X_1 < C$, so $\mathbb{E}[X_1] < C$, a contradiction since $\mathbb{E}[X_1] = \infty$.

January 2021

1.

Let μ be a probability measure on $\mathcal{B}([0,\infty))$ with the following property:

$$\mu([a,b]) = e^{-a} - e^{-b}, \text{ for all } 0 \leq a < b.$$

Show that μ is absolutely continuous with respect to the lebesgue measure from first principles.

Solution. Let $\tilde{\mu}$ be a measure defined by $\tilde{\mu}(A) = \int_A e^{-x}\ \lambda(dx)\ \forall\ A \in \mathcal{B}([0,\infty))$. $\tilde{\mu} << \lambda$ and $\tilde{\mu}([a,b]) = e^{-a} - e^{-b} = \mu([a,b])\ \forall\ 0 \leq a < b$. The set $\{[a,b] : 0 \leq a < b\}$ is a $\pi-$system hat generates $\mathcal{B}([0,1])$. Since $\tilde{\mu}$ and μ agree on this π system, by the $\pi - \lambda$ theorem, $\mu = \tilde{\mu}$ on $\mathcal{B}([0,\infty))$. Thus, $\mu << \lambda$.

2.

Let Y be a standard normal random variable, and let X be a random variable such that both pairs (X, Y) and $(X, X-Y)$ are independent. Show that X is constant with probability 1.

Solution. Since $(X, X-Y)$ and (X, Y) are independent, $\text{Cov}(X, X-Y) = \text{Cov}(X, Y) = 0$. Thus,

$$\begin{aligned}\text{Cov}(X, X-Y) &= \mathbb{E}[X(X-Y)] - \mathbb{E}[X]\mathbb{E}[X-Y] = \mathbb{E}[X^2 - XY] - \mathbb{E}[X]^2 + \mathbb{E}[X]\mathbb{E}[Y] \\ &= (\mathbb{E}[X^2] - \mathbb{E}[X]^2) - (\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]) = \text{Var}(X) - \text{Cov}(X, Y) = \text{Var}(X) = 0.\end{aligned}$$

Since $\text{Var}(X) = 0$. X is constant with probability 1.

3.

Let $\{X_n\}$ be a simple symmetrc random walk and let $|X| = M + A$ be the Doob-Meyer decomposition of the submartingale $|X|$, with respect to filtration generated by X, into martingale M with $M_0 = 0$ and a non-decreasing predictable process A. Show that M admits the representation

$$M = H \cdot X,$$

for some predictable process H and find the explicit expression for H.

Solution. For $|X| = M + A$, $A = \sum_{k=1}^{n} \mathbb{E}[|X_k| - |X_{k-1}||\mathcal{F}_{k-1}]$

$$\begin{aligned}&\text{For } X_{k-1} < 0,\ X_k \leq 0 \Rightarrow |X_k| - |X_{k-1}| = -X_k + X_{k-1} = -\xi_k \\ &\text{For } X_{k-1} > 0,\ X_k \geq 0 \Rightarrow |X_k| - |X_{k-1}| = X_k - X_{k-1} = \xi_k \\ &\text{For } X_{k-1} = 0\ |X_k| - |X_{k-1}| = |X_k| = 1 \\ &\text{So, } |X_k| - |X_{k-1}| = \xi_k(\mathbf{1}_{X_{k-1}>0} - \mathbf{1}_{X_{k-1}<0}) + \mathbf{1}_{X_{k-1}=0}\end{aligned}$$

Thus,

$$\begin{aligned}M_n = |X|_n - A_n &= |X_n| - \sum_{k=1}^{n} \mathbb{E}[\xi_k(\mathbf{1}_{X_{k-1}>0} - \mathbf{1}_{X_{k-1}<0}) + \mathbf{1}_{X_{k-1}=0}|\mathcal{F}_{k-1}] = |X_n| - \sum_{k=1}^{n} \mathbf{1}_{X_{k-1}=0} \\ &= \sum_{k=1}^{n}(|X_k| - |X_{k-1}|) - \sum_{k=1}^{n} \mathbf{1}_{X_{k-1}=0} = \sum_{k=1}^{n} \xi_k(\mathbf{1}_{X_{k-1}>0} - \mathbf{1}_{X_{k-1}<0}) + \mathbf{1}_{X_{k-1}=0} - \mathbf{1}_{X_{k-1}=0} \\ &= \sum_{k=1}^{n} \xi_k(\mathbf{1}_{X_{k-1}>0} - \mathbf{1}_{X_{k-1}<0}) = \sum_{k=1}^{n}(\mathbf{1}_{X_{k-1}>0} - \mathbf{1}_{X_{k-1}<0})(X_k - X_{k-1}) = (H \cdot X)_n\end{aligned}$$

where $H_k = \mathbf{1}_{X_{k-1}>0} - \mathbf{1}_{X_{k-1}<0}$ is a predictable process.

August 2021

1.

Let X_n be a sequence of random variables taking values in $\mathbb{N}$. Is it true that X_n converges a.s. if and only if X_n converges in probability? If it is, give a proof. Otherwise, give a counterexample.

Solution. In general, it true that $X_n \to X$ a.s. implies convergence in probability. It remains to show the other direction for X_n taking values in $\mathbb{N}$. Suppose X_n converges to X in probability. Then there exists a subsequence $\{X_{n_k}\}$ which converges to X a.s. Since $\{X_{n_k}\}$ is integer-valued, it only converges if it stabilizes. Thus, $X \in \mathbb{N}$. Since X_n, X are integer-valued, if $X_n \neq X$, $|X_n - X| \geq 1$. Let $\epsilon > 0$ be given. Thus, there exists an N such that for all $n \geq N$ since $\mathbb{P}[|X_n - X| \geq \frac{1}{2}] \leq \epsilon$ which implies $|X_n - X| = 0$ except on a set of measure at most ϵ. Taking, $\epsilon \to 0$, $X_n \to X$ a.s.

2.

Let $X_1, X_2, \dots$ be i.i.d random variables with values in $\mathbb{Z}^2$, where X_1 is uniformly distributed in $\{(k, m) : k \in \{-1, 0, 1\}, m \in \{-1, 0, 1\}\}$. Let $S_n = \sum_{i=1}^{n} X_i \in \mathbb{Z}^2$. Show that $\frac{S_n}{\sqrt{n}} \xrightarrow{d} S^*$, and find the distribution of S^*.

Solution. $\text{Var}(X_1) = \mathbb{E}[X_1^2] - \mathbb{E}[X_1]^2 = \mathbb{E}[X_1^2] = \left(\frac{2}{3}, \frac{2}{3}\right)$. Then, by the CLT,

$$\frac{S_n}{\sqrt{n}} \xrightarrow{d} \chi \sim N((0,0), (\frac{2}{3}, \frac{2}{3})).$$

3.

Give an example of a submartingale $\{X_n\}$ with the property that $X_n \to -\infty$ and $\mathbb{E}[X_n] \to +\infty$, as $n \to \infty$.

Solution.

August 2020

0.1 1.

Let X be a nonnegative random variable. Show that

$$\mathbb{E}[X\log^+(X)] < \infty \Leftrightarrow \int_1^\infty \int_1^\infty \mathbb{P}[X > uv]dudv < \infty,$$

where $\log^+(x) = \max(\log(x), 0)$.

Solution.

$$\begin{aligned}\int_1^\infty \int_1^\infty \mathbb{P}[X > uv]dudv &= \int_1^\infty \int_1^\infty \frac{1}{v}\mathbb{P}[X > w]\mathbf{1}_{w\geq v}dwdv = \int_1^\infty \mathbb{P}[X > w]\int_1^w \frac{1}{v}dvdw \\ &= \int_1^\infty \mathbb{P}[X > w]\log(w)dw = \int_1^\infty \int_0^\infty \mathbf{1}_{X>w}(x)\log(w)\mathbb{P}(dx)dw \\ &= \int_1^\infty \int_1^x \log(w)\ dw\mathbb{P}(dx) = \int_1^\infty x\log(x) - x + 1d\mathbb{P}(x) = \mathbb{E}[(X\log(X) - X + 1)\mathbf{1}_{X>1}] \\ &= \mathbb{E}[X\log^+(X) - X + 1].\end{aligned}$$

Since $x\log(x)$ grows faster than x, $\mathbb{E}[X\log^+(X) - X + 1] < \infty \Leftrightarrow \mathbb{E}[X\log^+(X)] < \infty$.

3.

Let X be a bounded random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, let $\mathcal{G}$ be a sub-$\sigma-$algebra of $\mathcal{F}$, and let $\mathbb{Q}$ be a measure on $\mathcal{F}$, absolutely continuous with respect to $\mathbb{P}$. Is the following

$$\mathbb{E}^{\mathbb{Q}}[X|\mathcal{G}] = \mathbb{E}[\frac{d\mathbb{Q}}{d\mathbb{P}}X|\mathcal{G}] \text{ a.s.}$$

always true? If so, prove it. If not, fix the right-hand side without using any (conditional) expectations under $\mathbb{Q}$.

Solution. This is not true. Take $\mathcal{G} = \mathcal{F}$. Then $\mathbb{E}^{\mathbb{Q}}[X|\mathcal{F}] = X$ and $\mathbb{E}[\frac{d\mathbb{Q}}{d\mathbb{P}}X|\mathcal{F}] = \frac{d\mathbb{Q}}{d\mathbb{P}}X$. Let $\xi = \mathbb{E}^{\mathbb{Q}}[X|\mathcal{G}]$ and $\eta = \mathbb{E}[\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{G}]$. We want to show $\xi\eta = \mathbb{E}[\frac{d\mathbb{Q}}{d\mathbb{P}}x|\mathcal{G}]$. Since ξ, η are $\mathcal{G}$ measurable, $\xi\eta$ are $\mathcal{G}$ measurable. Let $A \in \mathcal{G}$. We want to show $\mathbb{E}[\xi\eta\mathbf{1}_A] = \mathbb{E}[x\frac{d\mathbb{Q}}{d\mathbb{P}}\mathbf{1}_A]$.

$$\mathbb{E}[\xi\frac{d\mathbb{Q}}{d\mathbb{P}}\mathbf{1}_A] = \int_A \xi\frac{d\mathbb{Q}}{d\mathbb{P}}d\mathbb{P} = \int_A \xi d\mathbb{Q} = \int_A xd\mathbb{Q} = \int_A x\frac{d\mathbb{Q}}{d\mathbb{P}}d\mathbb{P} = \mathbb{E}[X\frac{d\mathbb{Q}}{d\mathbb{P}}\mathbf{1}_A].$$

Thus, it is sufficent to show $\mathbb{E}[\xi\eta\mathbf{1}_A] = \mathbb{E}[\xi\frac{d\mathbb{Q}}{d\mathbb{P}}\mathbf{1}_A]$.

$$\mathbb{E}[\xi\eta\mathbf{1}_A] = \int_A \xi\eta d\mathbb{P} = \int_A \xi\mathbb{E}[\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{G}]d\mathbb{P} = \int_A \mathbb{E}[\xi\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{G}]d\mathbb{P} = \int_A \xi\frac{d\mathbb{Q}}{d\mathbb{P}}d\mathbb{P} = \mathbb{E}[\xi\frac{d\mathbb{Q}}{d\mathbb{P}}\mathbf{1}_A].$$

So, by the definition of conditional expectation,

$$\mathbb{E}^{\mathbb{Q}}[X|\mathcal{G}] = \frac{\mathbb{E}[\frac{d\mathbb{Q}}{d\mathbb{P}}X|\mathcal{G}]}{\mathbb{E}[\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{G}]}.$$

January 2019

3.

Let Z_1 and Z_2 be independent standard normals. Find the conditional density of $e^{3Z_1+Z_2}$ given $\sigma(e^{Z_1+2Z_2})$.

Solution. Let $X = 3Z_1 + Z_2$, $Y = Z_1 + 2Z_2$, $Z = 2Z_1 - Z_2$. Since Z_1 and Z_2 are independent standard normals, X, Y, and Z are normal random variables. Since normal random variables are independent if and only if they are uncorrelated, and

$$\text{Cov}(Y, Z) = \mathbb{E}[YZ] = \mathbb{E}[2Z_1^2 - Z_1Z_2 + 4Z_1Z_2 - 2Z_2^2] = 0,$$

Y and Z are independent. Further, we can write $X = Y + Z$. Thus, when we condition X on Y, as this would be equivalent to conditioning on e^Y, X conditioned on Y is a normal random variable with mean Y and variance $\text{Var}(X|Y) = \text{Var}(Z) = 5$. Since the normal distribution for a random variable with mean μ and variance σ^2 is given by

$$\frac{1}{\sqrt{2\pi\sigma^2}}\exp(-\frac{(x-\mu)^2}{2\sigma^2}).$$

Thus, the distribution its exponential function is

$$\frac{1}{x\sqrt{2\pi\sigma^2}}\exp(-\frac{(\log(x)-\mu)^2}{2\sigma^2}).$$

So, the conditional density of e^X given Y is

$$\frac{1}{x\sqrt{10\pi}}\exp(-\frac{(\log(x)-Y)^2}{10}).$$

January 2018

3.

Let X and Y be random variables in $\mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ such that either

1. $(X(\omega') - X(\omega))\,(Y(\omega') - Y(\omega)) \geq 0 \;\forall\; \omega, \omega' \in \omega$
2. the function $y \mapsto \mathbb{E}[X|Y = y]$ is nondecreasing.

Show $\mathrm{Cov}(X, Y) \geq 0$.

Solution.

1. Suppose $(X(\omega') - X(\omega))\,(Y(\omega') - Y(\omega)) \geq 0 \;\forall\; \omega, \omega' \in \Omega$.

$$Cov(X,Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = \int_\Omega (X - \mathbb{E}[X])(Y - \mathbb{E}[Y]) d\mathbb{P}$$
$$= \int_\Omega \int_\Omega (X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) d\mathbb{P}(\omega') d\mathbb{P}(\omega) \geq 0$$

2. Suppose $y \mapsto \mathbb{E}[X|Y = y]$ is nondecreasing.

August 2018

3.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\mathcal{G}$ a sub-σ-algebra of $\mathcal{F}$ and $\{A_n\}$ a sequence of $\mathcal{G}$ independent random variables. Show that
$$\left\{ \sum_{i=1}^\infty \mathbb{P}[A_i|\mathcal{G}] = \infty \right\} = \left\{ \mathbb{P}[\limsup_{i\to\infty} A_i|\mathcal{G}] = 1 \right\}, \text{ a.s.,}$$
where, as usual two events are equal a.s., if their indications are a.s.-equal random variables.

Solution. We are given the following equivalent definition of conditional independence: $\{A_i\}$ is an independent sequence under the probability measure $\mathbb{P}_B := \mathbb{P}[\cdot \cap B]/\mathbb{P}[B]$ for each $B \in \mathcal{G}$ with $\mathbb{P}[B] > 0$.

Call the left and right sides of the equation L and R respectively. Then $L, R \in \mathcal{G}$. For $B \in \mathcal{G}$ with $\mathbb{P}[B \cap L] > 0$, Fubini's theorem and the definition of conditional expectation imply

$$\infty = \mathbb{E}_{B\cap L}\left[\sum_i \mathbb{P}[A_i|\mathcal{G}] \right] = \int \sum_i \mathbb{P}[A_i|\mathcal{G}] d\mathbb{P}_{L\cap B} = \frac{1}{\mathbb{P}(B\cap L)} \int \sum_i \mathbb{P}[A_i|\mathcal{G}] \mathbf{1}_{B\cap L} d\mathbb{P}$$
$$= \frac{1}{\mathbb{P}[B\cap L]} \sum_i \mathbb{E}\left[\mathbb{P}[A_i|\mathcal{G}] \mathbf{1}_{B\cap L} \right] = \frac{1}{\mathbb{P}[L\cap B]} \sum_i \mathbb{P}[A_i \cap (L\cap B)] = \sum_i \mathbb{P}_{L\cap B}[A_i]$$

By the given equivalent definition of conditional independence, $\{A_i\}$ are independent under $\mathbb{P}_{L\cap B}$. Thus, by the second Borel-Cantelli Lemma,

$$\mathbb{P}_{B\cap L}[\limsup_i A_i] = \frac{\mathbb{E}[\mathbf{1}_B \mathbf{1}_L \mathbf{1}_{\limsup_i A_i}]}{\mathbb{E}[\mathbf{1}_B \mathbf{1}_L]} = 1 \to \mathbb{E}[\mathbf{1}_B \mathbf{1}_L \mathbf{1}_{\limsup_i A_i}] = \mathbb{E}[\mathbf{1}_B \mathbf{1}_L] \;\forall\; B \in \mathcal{G}.$$

This equality is satisfied trivially if $\mathbb{P}[B \cap L] = 0$.

By the definition of conditional expectation,

$$\mathbb{E}[\mathbf{1}_B \mathbf{1}_L \mathbf{1}_{\limsup_i A_i}] = \mathbb{E}[(\mathbf{1}_{\limsup_i A_i} \mathbf{1}_L) \mathbf{1}_B] = \mathbb{E}[1_L \mathbf{1}_B]$$

so by the definition of conditional expectation,

$$\mathbb{E}[\mathbf{1}_{\limsup_i A_i} \mathbf{1}_L | \mathcal{G}] = \mathbf{1}_L \mathbb{P}[\limsup_i A_i|\mathcal{G}] = \mathbf{1}_L \text{ a.s.}$$

So $\mathbb{P}[\limsup_i A_i] = 1$ a.s. which implies $L \subset \mathbb{R}$ a.s.

For the other inclusion, define $L_n^c = \{\sum_i \mathbb{P}[A_i|\mathcal{G}] \leq n\} \in \mathcal{G}$. For $\mathbb{P}[L_n^x \cap B] > 0$, we have

$$\infty > \frac{n}{\mathbb{P}[L_n^c \cap B]} \geq \mathbb{E}_{B\cap L_n^c}\left[\sum_i \mathbb{P}[A_i|\mathcal{G}] \right] = \sum_i \mathbb{P}_{B\cap L_n^c}[A_i].$$

Thus, by the first Borel-Cantelli Lemma, $\mathbb{P}_{B\cap L_n^c}[\limsup_i A_i] \Rightarrow \mathbb{E}[\mathbf{1}_B \mathbf{1}_{L_n^c} \mathbf{1}_{\limsup_i A_i}] = 0$. So, by the definition of conditional expectation, $\mathbf{1}_{L_n^c} \mathbb{P}[\limsup_i A_i|\mathcal{G}] = 0$ a.s. That is $L_n^c \subset R^c$, a.s. It is clear that $L_n^c \subset L^c$. Let $\omega \in L^c$, then since $\mathbb{P}[A_i|\mathcal{G}](\omega) \geq 0 \;\forall\; i$ and $\sum_i \mathbb{P}[A_i|\mathcal{G}] \neq \infty$, there exists an N such that $\sum_i \mathbb{P}[A_i|\mathcal{G}](\omega) < N$, so $\omega \in L_N^c \subset \bigcup_n L_n^c$, so $L^c = \bigcup_n L_n^c$. Thus, $R^c = L^c$, so $R = L$.

January 2016

1.

Let $X_1, \dots, X_n$, $n \geq 2$ be iid absolutely continuous random variables, with density f. Consider the random vector $X = (X^{(1)}, X^{(n)})$, where

$$X^{(1)} = \min(X_1, \dots, X_n) \text{ and } X^{(n)} = \max(X_1, \dots, X_n).$$

1. Derive the joint density f_X of X and the density of the range $X^{(n)} - X^{(1)}$.
2. n iid points are chosen uniformly in the square $[0,1]^2$. Let A be the area of the smallest rectangle with sides parallel to the sides of the square $[0,1]^2$, which contains all n points. Compute the moments $\mathbb{E}[A^k]$, $k \in \mathbb{N}$, of A.

Solution.

1. First, we want to derive the joint density f_X.

$$\begin{aligned}
\mathbb{P}[X^{(1)} \leq x, X^{(n)} \leq y] &= \mathbb{P}[(\bigcup_{i=1}^n \{X_i \leq x\}) \cap (\bigcap_{i=1}^n \{X_i \leq y\})] = \mathbb{P}[(\bigcap_{i=1}^n \{X_i \leq y\}) \setminus (\bigcap_{i=1}^n \{x < X_i \leq y\})] \\
&= \mathbb{P}[\bigcap_{i=1}^n \{X_i \leq y\}] - \mathbb{P}[\bigcap_{i=1}^n \{x < X_i \leq y\}] = \mathbb{P}[X_1 \leq y]^n - \mathbb{P}[x < X_1 \leq y]^n \\
&= \left(\int_{-\infty}^y f(x)dx\right)^n - \left(\int_x^y f(x)dx\right)^n \\
\Rightarrow f_X(x,y) &= n(n-1)f(x)f(y)\mathbb{P}[x < X_1 \leq y]^{n-2}
\end{aligned}$$

2.

Let μ be a probability measure on $\mathbb{R}$. Show that the following are equivalent, where φ_μ denotes the characteristic function of μ:

1. μ is supported by a set of the form $\{an + b : n \in \mathbb{Z}\}$ for a pair of rational numbers a, b.
2. $\varphi_\mu(2\pi t_0) = 1$ for some rational $t_0 \neq 0$.

Solution. (1) $\to$ (2). Suppose μ is supported on $\{\frac{a}{b}n + \frac{c}{d} : n \in \mathbb{N}\}$ for $a, b, c, d \in \mathbb{Z}$ and $b, d \neq 0$. Then

$$\varphi_\mu(t) = \int_\infty^\infty e^{itx}\mu(dx) = \sum_{n \in \mathbb{N}} e^{it(\frac{a}{b}n + \frac{c}{d})}\mu(\{\frac{a}{b}n + \frac{c}{d}\}) = \sum_{n \in \mathbb{N}} (\cos(t(\frac{a}{b}n + \frac{c}{d})) + i\sin(t(\frac{a}{b}n + \frac{c}{d})))\mu(\{an + b\}).$$

Thus, it is sufficient to find t_0 such that $\cos(2\pi t_0(\frac{a}{b}n + \frac{c}{d}) = \cos(2\pi \frac{t_0}{bd}(adn + cb)) = 1 \; \forall \; n$. Since $abn + cd \in \mathbb{Z}$, for $t_0 = bd$, $\cos(2\pi \frac{t_0}{bd}(adn + bc)) = 0$.

(2) $\to$ (1). Suppose $\phi_\mu(2\pi t_0) = 1$ for some rational $t_0 \neq 0$.

$$1 = \phi_\mu(2\pi t_0) = \int_{-\infty}^\infty e^{i(2\pi t_0)x}\mu(dx) \leq \int_{-\infty}^\infty \mu(dx) = 1,$$

$e^{i(2\pi t_0)x} = 1$ on the support of μ. Thus, μ is supported of x such that $\cos(2\pi t_0 x) = 1$, that is when $t_0 x \in \mathbb{Z}$. Since t_0 is rational, we can express $t_0 = \frac{a}{b}$. Then $\frac{a}{b}x = n \in \mathbb{Z}$, so $x = \frac{b}{a}n$. So, μ is supported on the set $\{\frac{1}{t_0}n : n \in \mathbb{Z}\}$.

3.

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a random variable $X \in \mathbb{L}^2(\mathcal{F})$ and a sub$-\sigma$-algebra $\mathcal{G} \subset \mathcal{F}$. Find the projection of X on the space $\mathbb{L}^2(\mathcal{G})$ of square-integrable random variables measurable with respect to $\mathcal{G}$. In other words, find the random variable $\hat{Y}$ that attains

$$\min_{Y \in \mathbb{L}^2} \mathbb{E}[|X - Y|^2].$$

Justify your answers.

Solution. Since $\mathbb{L}^2(\mathcal{F})$ is a Hilbert space and $\mathbb{L}^2(\mathcal{G})$ is a closed subset of $\mathbb{L}^2(\mathcal{F})$, there exists a unique $\hat{Y} \in \mathbb{L}^2(\mathcal{G})$ such that $\mathbb{E}[|X - \hat{Y}|^2] = \min_{Y \in \mathbb{L}^2(\mathcal{G})} \mathbb{E}[|X - Y|^2]$. Thus, we know $\hat{Y}$ is $\mathcal{G}-$measurable. Further, since $\hat{Y}$ minimizes, we have $\mathbb{E}[(X - \hat{Y})(Z - \hat{Y})] = 0 \; \forall \; Z \in \mathbb{L}^2(\mathcal{G})$. Choose $Z = \hat{Y} - \mathbf{1}_A$ for $A \in \mathcal{G}$. Then $\mathbb{E}[(X - \hat{Y})\mathbf{1}_A] = 0 \Rightarrow \mathbb{E}[X\mathbf{1}_A] = \mathbb{E}[\hat{Y}\mathbf{1}_A]$. So $\hat{Y} = \mathbb{E}[X|\mathcal{G}]$.