# August 2013

## 1.

Let $\{B_t\}$ be the Brownian motion, and let $\sigma$ be the last visit to the level 0 before $t = 1$, i.e.,

$$\sigma = \sup\{t \leq 1 : B_t = 0\}.$$

1. Show that $\sigma$ is not a stopping time

2. Show that
$$\sigma \overset{d}{=} \frac{X^2}{X^2 + Y^2}$$
   where $X$ and $Y$ are independent unit normals.

**Solution.**

1. Since $B_t$ is BM, $B_t^2 - t$ is a martingale. For the sake of contradiction, suppose $\sigma$ is a stopping time. Then since $\sigma$ is bounded, we may apply the Bounded optional sampling theorem,

$$\mathbb{E}[B_\sigma^2 - \sigma] = 0 \Rightarrow \mathbb{E}[\sigma] = 0.$$

This, implies $\mathbb{P}[\sigma = 0] = 1$. However, we know that $\sigma$ has an arcsine distribution, so

$$\mathbb{P}[\sigma = 0] = \frac{2}{\pi} \arcsin(0) - 0,$$

a contradiction,

2. $\{\sigma \leq t\} = \{\sup_{t \leq s \leq 1} B_s < 0\} \cup \{\inf_{t \leq s \leq 1} B_s > 0\}$, so since $(\sup_{t \leq s \leq 1} B_s - B_t, B_t) \sim (|B_1 - B_t|, B_t)$ and $|B_1 - B_t|$ and $B_t$ are independent.

$$\mathbb{P}[\sigma \leq t] = 2\mathbb{P}[\sup_{t \leq s \leq 1} B_s < 0] = 2\mathbb{P}[\sup_{t \leq s \leq 1}(B_s - B_t) \leq -B_t] = 2\mathbb{P}[M_{1-t} < -B_t] = 2\mathbb{P}[|B_1 - B_t| < -B_t]$$
$$= \mathbb{P}[|B_1 - B_t| < |B_t|] = \mathbb{P}[(B_1 - B_t)^2 < B_t^2]$$

Since $B_1 - B_t \sim N(0, (1-t)) \sim \sqrt{1-t}X$, and $B_t \sim N(0,t) \sim \sqrt{t}Y$ where $X, Y \sim N(0,1)$. Then

$$\mathbb{P}[\sigma \leq t] = \mathbb{P}[(1-t)X^2 < tY^2] = \mathbb{P}[\frac{X^2}{X^2 + Y^2} < t].$$

so $\sigma \sim \frac{X^2}{X^2+Y^2}$.

## 2.

Let $M$ and $N$ be two continuous local martingales. Show that $\mathcal{E}(M)\mathcal{E}(N)$ is a local martingale if and only if $MN$ is a local martingale.

**Solution.** By the definition of quadratic covariation, $MN$ is a local martingale if and only if $\langle M, N \rangle = 0$. Further, $\mathcal{E}(M)\mathcal{E}(N) = \mathcal{E}(M+N)\mathcal{E}(\langle M, N \rangle) = \mathcal{E}(M+N)\exp(\langle M, N \rangle)$. Since $M+N$ is a local martingale, $\mathcal{E}$ is a local martingale. Thus, if $MN$ is a local martingale, $\langle M, N \rangle = 0$, so $\mathcal{E}(M)\mathcal{E}(N) = \mathcal{E}(M+N)e^0 = \mathcal{E}(M+N)$, so $\mathcal{E}(M)\mathcal{E}(N)$ is a local martingale.

$$\langle \mathcal{E}(M)\mathcal{E}(N) \rangle = \int_0^t \mathcal{E}(M)\mathcal{E}(N)d\langle M, N \rangle = 0$$

Since $\mathcal{E}(M)\mathcal{E}(N) > 0$, $\langle M, N \rangle$ must be constant (?). So, since $\langle M, N \rangle_0 = 0$, $\langle M, N \rangle = 0$, so $MN$ is a local martingale.

## 3.

Let $\{B_t\}$ be a Brownian motion. For $c \in \mathbb{R}$, compute

$$\mathbb{P}[B_t + ct < 1, \forall \, t \geq 0].$$

**Solution.** Let $X_t = B_t + ct$. For $c > 0$, $\mathbb{P}[B_t + ct < 1, \forall \, t \geq 0] = 0$. Thus, assume $c \leq 0$. For Brownian motion $B_t$, we know that the exponential martingale $\exp(\lambda B_t - \frac{1}{2}\lambda^2 t)$ is a martingale for $\lambda \in \mathbb{R}$. Then, we can choose $\lambda = -2c$, so that $M_t = \exp(-2c(B_t + ct)) = \exp(-2cX_t)$. Since $c < 0$, $M_t$ is a martingale such that $M_t \to 0$ and $t \to \infty$. Define $\tau = \inf\{t \geq 0 : M_t = \exp(-2c)\}$. Since $M_t^\tau$ is bounded and therefore uniformly integrable, by the optional sampling theorem

$$\mathbb{E}[M_\tau] = \mathbb{P}[\tau < \infty]\exp(-2c) + \mathbb{P}[\tau = \infty](0) = \mathbb{E}[M_0] = 1 \to \mathbb{P}[\tau < \infty] = \exp(2c)$$

$$\mathbb{P}[X_t < 1 \, \forall \, t \geq 0] = \mathbb{P}[\sup_{t \geq 0} X_t < 1] = \mathbb{P}[\sup_{t \geq 0} M_t < \exp(-2c)] = 1 - \mathbb{P}[\tau < \infty] = 1 - \exp(2c).$$

# January 2014

## 1.

Let $(M_t)_{0 \le t \le T}$ be a submartingale and let $\lambda > 0$. Show that

$$\lambda \mathbb{P}(\max_{0 \le t \le T} M_t \ge \lambda) \le \mathbb{E}[M_t \mathbf{1}_{\max_{0 \le t \le T} M_t \ge \lambda}].$$

1. Consider a one-dimensional Brownian motion at $B_0 = 0$. Let $u, v : [0, \infty) \to \mathbb{R}$ be such that $u$ is $C^1$, strictly increasing and $u(0) = 0$. Assume also that $v(t) \ne 0$ for each $t$ and $v$ has bounded variation. Show that the process

$$X_t = v(t) B_{u(t)}$$

is a semi-martingale and the martingale part is $\int_0^t v(s) dB_{u(s)}$.

2. Show that the martingale part is a Brownian motion if and only if $v^2(s) u'(s) = 1$ for each $s$

3. Find $u, v$ such that $X$ defined above is an Orstein-Uhlenbeck process with parameter $\beta$, $dX_t = \beta X_t dt + d\gamma_t$ for some Brownian motion $\gamma$.

**Solution.** Let $(M_t)_{0 \le t \le T}$ be a submartingale. Then

$$\lambda \mathbb{P}[\max_{0 \le t \le T} M_t \ge \lambda] = \mathbb{E}[\lambda \mathbf{1}_{\max_{0 \le t \le T} M_t \ge \lambda}] \le \mathbb{E}[M_t \mathbf{1}_{\max_{0 \le t \le T} M_t \ge \lambda}].$$

1. First, let's show that $M_t = \int_0^t v(s) dB_{u(s)}$ is a martingale. Let $r < t$. Let $\{\Delta_n\}$ be a sequence of partitions such that $\Delta_n \to \text{Id}$. Then let $M_t^n = \sum_{k=1}^n v(t_k)(B_{t_k \wedge t} - B_{t_{k-1} \wedge t})$. So, $\mathbb{E}[M_t^n | \mathcal{F}_r] = \sum_{k=1}^n v(t_k) \mathbb{E}[(B_{t_k \wedge t} - B_{t_{k-1} \wedge t}) | \mathcal{F}_r] = \sum_{k=1}^n v(t_k)(B_{t_k \wedge r} - B_{t_{k-1} \wedge r}) = M_r^n$. Taking the limit as $n \to \infty$, $M_t$ is a martingale. Now, we want to show that $X_t - \int_0^t v(s) dB_{u(s)}$ is of finite variation. To do this we look at the total variation,

$$\sup \sum_{i=1}^k |v(t_i) B_{u(t_i)} - \int_0^{t_i} v(s) dB_{u(s)} - v(t_{i-1}) B_{u(t_{i-1})} + \int_0^{t_{i-1}} v(s) dB_{u(s)}|$$

$$= \sup \sum_{i=1}^k |(v(t_i) B_{u(t_i)} - v(t_{i-1}) B_{u(t_{i-1})}) + \int_{t_{i-1}}^{t_i} v(s) dB_{u(s)}|$$

2. By the Lévy characterization of a Brownian motion, we just need $\langle \int_0^t v(s) dB_{u(s)} \rangle = t$. Let $M_t = \int_0^t v(s) dB_{u(s)}$. Then $d\langle M_t \rangle = dM_t dM_t = v(t)^2 dB_{u(t)} dB_{u(t)} = v(t)^2 du(t)$, so $\langle M \rangle_t = \int_0^t v(s)^2 du(s) = \int_0^t v(s)^2 u'(s) ds = t$. Since $v^2(s), u'(s) \ge 0 \ \forall \ s \in [0, \infty)$, $v(s)^2 u'(s) = 1 \ \forall \ s$.

3. Since $X_t$ is a semimartingale, we can apply Ito's formula,

$$dX_t = v(t) dB_{u(t)} + B_{u(t)} dv(t).$$

To be a OU process, need $\int_0^t v(s) dB_{u(s)}$ to be a Brownian motion, which implies $v^2 u' = 1$ from above and $B_{u(t)} dv(t) = \beta v(t) B_{u(t)} dt$. This occurs when $dv(t) = \beta v(t) dt$, so $v(t) = C e^{\beta t}$, $C \ne 0$. Then since $v^2 u' = 1$, implies $u'(t) = C^{-2} e^{-2\beta t}$. Thus, $u(t) = -\frac{1}{2C^2 \beta} e^{-2\beta t} + D$. Since $u(0) = 0$, $D = \frac{1}{2C^2 \beta}$, so $u(t) = \frac{1}{2C^2 \beta}(1 - e^{-2\beta t})$.

## 2.

(Range of Brownian Motion) Let $B$ be a one-dimensional BM starting at zero. Define

$$S_t = \max_{s \le t} B_s, \quad I_t = \inf_{s \le t} B_s, \quad \theta_c = \inf\{t : S_t - I_t = c\},$$

for some $c > 0$.

1. Show that for each $\lambda$, the process

$$M_t = \cosh(\lambda(S_t - B_t)) \exp(-\frac{\lambda^2 t}{2})$$

is a martingale.

2. Prove that $\mathbb{E}[\exp(-\frac{\lambda^2 \theta_c}{2})] = \frac{2}{1 + \cosh(\lambda c)}$.

**Solution**

1. Let $0 \leq s < t$.

$$\mathbb{E}[\exp(\lambda(S_t - B_t))|\mathcal{F}_s] = \mathbb{E}[\exp(\lambda((S_t - B_t) - (S_s - B_s) + (S_s - B_s))|\mathcal{F}_s]$$
$$= \exp(\lambda(S_s - B_s))\mathbb{E}[\exp(\lambda((S_t - S_s) - (B_t - B_s))|\mathcal{F}_s]$$

$$\mathbb{E}[\exp(\lambda((S_t - S_s) - (B_t - B_s))|\mathcal{F}_s] = \mathbb{E}[\exp(\lambda(|B_t - B_s|)] = \frac{2}{\sqrt{2\pi}}\int_0^\infty e^{\lambda x} e^{-\frac{x^2}{2(t-s)}} dx$$
$$= \frac{2}{\sqrt{2\pi(t-s)}} e^{\frac{\lambda^2(t-s)}{2}} 2\int_0^\infty e^{-\frac{x^2}{2(t-s)}+\lambda x - \frac{\lambda^2(t-s)}{2}} dx$$
$$= \frac{2}{\sqrt{2\pi(t-s)}} e^{\frac{\lambda^2(t-s)}{2}} \int_0^\infty e^{-(\frac{x}{\sqrt{2(t-s)}} - \frac{\lambda\sqrt{t-s}}{\sqrt{2}})^2} dx$$
$$= \frac{2}{\sqrt{2\pi(t-s)}} e^{\frac{\lambda^2(t-s)}{2}} \sqrt{2(t-s)} \int_0^\infty e^{-u^2} du$$
$$= \frac{2}{\sqrt{\pi}} e^{\frac{\lambda^2(t-s)}{2}} \frac{\sqrt{\pi}}{2} = e^{\frac{\lambda^2(t-s)}{2}}.$$

Similarly, $\mathbb{E}[\exp(-\lambda(S_t - B_t))|\mathcal{F}_s] = \exp(\lambda(S_s - B_s))e^{\frac{\lambda^2(t-s)}{2}}$. Thus,

$$\mathbb{E}[\cosh(\lambda(S_t - B_t))|\mathcal{F}_s] = \cosh(\lambda(S_s - B_s))e^{\frac{\lambda^2(t-s)}{2}}$$

which implies

$$\mathbb{E}[M_t|\mathcal{F}_s] = \cosh(\lambda(S_s - B_s))e^{\frac{\lambda^2(t-s)}{2}} e^{\frac{-\lambda^2 t}{2}} = M_s.$$

So $M_t$ is a martingale.

2. The stopped martingale $M_t^{\theta_c}$ is bounded and thus uniformly integrable. So, by the optional sampling theorem,

$$\mathbb{E}[M_{\theta_c}] = \mathbb{E}[\cosh(\lambda(S_{\theta_c} - B_{\theta_c})\exp(-\frac{\lambda^2\theta_c}{2}] = \mathbb{E}[M_0] = 1$$
$$\Rightarrow \mathbb{E}[\exp(-\frac{\lambda^2\theta_c}{2}] = \frac{1}{\mathbb{E}[\cosh(\lambda(S_{\theta_c} - B_{\theta_c}))]}$$

Since $B_{\theta_c} = S_{\theta_c}$ or $B_{\theta_c} = I_{\theta_c}$ with equal probability (by the symmetry of BM),

$$\mathbb{E}[\cosh(\lambda(S_{\theta_c} - B_{\theta_c}))] = \frac{1}{2} + \frac{1}{2}\cosh(\lambda c).$$

So,

$$\mathbb{E}[\exp(-\frac{\lambda^2\theta_c}{2})] = \frac{2}{1 + \cosh(\lambda c)}.$$

# August 2014

## 2.

Let $(W_t)_{0 \leq t \leq 1}$ a BM (defined only up to time one). Show that the two dimensional vector

$$\left(W_1, \int_0^1 sgn(W_s)dW_s\right)$$

has the following properties:

1. both marginals are normal $N(0,1)$

2. however, it is NOT a joint normal random vector

**Solution.**

1. From the definition of BM, $W_1 = W_1 - W_1 \sim N(0,1)$. Let $\Delta_n$ be a sequence of partitions of $[0,1]$ such that $\Delta_n \to Id$. Then $M_t^n = \sum_{i=1}^n sgn(W_{t_i})(W_{t_i} - W_{t_{i-1}})$. From the definition of BM, these increments $W_{t_i} - W_{t_{i-1}}$ are independent and $sgn(W_{t_i})(W_{t_i} - W_{t_{i-1}}) \sim N(0, t_i - t_{i-1}$. Thus, since $\Delta_n$ is a partition of $[0,1]$, $M_t^n \sim N(0,1) \ \forall \ n$. Thus, since $M_t^n \to M_t$ as $n \to \infty$, $M_t \sim N(0,1)$.

2. 

## 3.

(Stratonovich integral and chain rule) For two continuous semi-martingales $X$ and $Y$ (on the same space and filtration), we define the Stratonovich integral

$$\int_0^t X_s \circ dY_s = \int_0^t X_s dY_s + \frac{1}{2}\langle X, Y \rangle_t,$$

where $\int_0^t X_s dY_s$ represents the Itô integral. Show that if $f \in C^3$, and $X$ is a continuous semimartingale, then we have the chain rule

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) \circ dX_s.$$

**Solution.** Since $f \in C^3$ and $X$ is a a continuous semi-martingale, we can apply Itô's formula,

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) dX_s + \frac{1}{2} \int_0^t f''(X_s) d\langle X \rangle_s.$$

Since $\int_0^t f'(X_s) \circ dX_s = \int_0^t f'(X_s) dX_s + \frac{1}{2}\langle f'(X), X \rangle_t$, it is sufficient to show that

$$\int_0^t f''(X_s) d\langle X \rangle_s = \langle f'(X), X \rangle_t.$$

Since $f \in C^3, f' \in C^2$, so we can apply Itô again,

$$d\langle f'(X), X \rangle_t = df'(X_t) dX_t = (f''(X_t) dX_t + \frac{1}{2} f'''(X_t) d\langle X \rangle_t) dX_t = f''(X_t) d\langle X \rangle_t + \frac{1}{2} f'''(X_t) d\langle X \rangle_t dX_t = f''(X_t) d\langle X \rangle_t$$

So,

$$\int_0^t f''(X_s) d\langle X \rangle_s = \langle f'(X), X \rangle_t.$$

# August 2015

### 1.

Consider two pairs of adapted continuous processes $(H^i, X^i)$ defined on two filtered probability spaces, $(\Omega_i, \mathcal{F}_i, (\mathcal{F}_t^i)_{0 \le t < \infty}, \mathbb{P}_i)$ for $i = 1, 2$. Assume that the two pairs have the same law (as two dimensional processes), and that $X^1, X^2$ are semi-martingalees. Show that the two stochastic integrals $I^i = \int H^i dX^i$ have the same law for $i = 1, 2$.

### 2.

Consider a binary random variable $X$ such that $\mathbb{E}[X] = 0$. For a given Brownian motion $B$, construct a stopping time with property $\mathbb{E}[T] < \infty$ and such that $B_T$ and $X$ have the same distribution. Is the condition $\mathbb{E}[X] = 0$ necessary for the existence of such a stopping time?

**Solution.** Let $X$ be a binary random variable. That is

$$\mathbb{P}[X = a] = 1 - \mathbb{P}[X = b]$$

for some $a \ne b \in \mathbb{R}$ such that

$$\mathbb{E}[X] = a\mathbb{P}[X = a] + b\mathbb{P}[X = b] = 0 \to a\mathbb{P}[X = a] + b(1 - \mathbb{P}[X = a]) \to \mathbb{P}[X = a] = \frac{b}{b-a}.$$

This implies one of $a, b$ is negative. WLOG, assume $a > 0, b < 0$. So,

$$X = \begin{cases} a & p = \frac{b}{b-a} \\ b & p = \frac{a}{b-a}. \end{cases}$$

Let $B_t$ be a Brownian motion. Consider the stopping time $\tau = T_a \wedge T_b$ where $T_a = \inf\{t \ge 0 : B_t = a\}$ and $T_b = \inf\{t \ge 0 : B_t = b\}$. Since $B_t^\tau$ is a bounded martingale and therefore uniformly integrable, we can apply the optional sampling theorem,

$$\mathbb{E}[B_\tau] = a\mathbb{P}[T_a < T_b] + b\mathbb{P}[T_b < T_a] = 0 \Rightarrow \mathbb{P}[T_a < T_b] = \frac{b}{b-a}.$$

So

$$B_\tau = \begin{cases} a & p = \frac{b}{b-a} \\ b & p = \frac{-a}{b-a}. \end{cases}$$

Since $B_t$ is BM, $B_t^2 - t$ is a martingale. For fixed $t$, we can apply the bounded optional sampling theorem so that

$$\mathbb{E}[B_{t \wedge \tau}^2] = \mathbb{E}[t \wedge \tau].$$

Taking the limit as $t \to \infty$ we get

$$\mathbb{E}[B_\tau^2] = \mathbb{E}[\tau]$$

by the bounded convergence theorem for the LHS and the monotone convergence theorem for the RHS. Thus,

$$\mathbb{E}[\tau] = a^2 \mathbb{P}[T_a < T_b] + b^2(1 - \mathbb{P}[T_a < T_b]) < a^2 + b^2 < \infty.$$

**3.**

Show that, for a continuous semimartingale $M$ and continuous adapted process $A$ of bounded variation, with $A_0 = 0$, we have the following equivalence:

1. $M$ is actually a local martingale with $\langle M \rangle = A$

2. for each $f \in C_b^2$, we have that

$$f(M_t) - f(M_0) - \int_0^t f''(M_s)dA_s$$

is a martingale.

**Solution.** Let $f \in C_b^2$. By Itô's formula,

$$f(M_t) = f(M_0) + \int_0^t f'(M_s)dM_s + \frac{1}{2}\int_0^t f''(M_s)d\langle M \rangle_s$$

$(1) \to (2)$ Suppose $M$ is a local martingale with $\langle M \rangle = A$. Then,

$$\int_0^t f'(M_s)dM_s = f(M_t) - f(M_0) - \frac{1}{2}\int_0^t f''(M_s)dA_s.$$

Since $M_t$ is a local martingale and $f'$ is bounded, $\int_0^t f'(M_s)dM_s$ is a local martingale. Further,

$$\langle \int_0^t f'(M_s)dM_s \rangle = \int_0^t f'(M_s)^2 dA_s \leq CA_t < \infty \ \forall t \geq 0$$

So, $f(M_t) - f(M_0) - \frac{1}{2}\int_0^t f''(M_s)dA_s$ is a martingale.
$(2) \to (1)$

# January & August 2016

**1.**

Let $S$ be an exponential Brownian motion with drift,

$$S_t = 1 + \int_0^t \mu S_u du + \int_0^t S_u dB_u, \quad t \geq 0,$$

for some $\mu \in \mathbb{R}$, where $B$ is the standard Brownian motion. Given $\epsilon \in (0,1)$, compute $\mathbb{E}[\tau_\epsilon]$, where $\tau_\epsilon = \inf\{t \geq 0 : S_t = \epsilon\}$.
   **Solution.**
$$S_t = \exp((\mu - 1)t + B_t)$$

We can check this using Itô's formula,

$$dS_t = S_t d((\mu - \frac{1}{2})t + B_t) + \frac{1}{2}S_t d\langle(\mu - \frac{1}{2})t + B_t\rangle = S_t((\mu - \frac{1}{2})dt + dB_t + \frac{1}{2}dt) = S_t(\mu dt + dB_t).$$

This is hitting time of Brownian motion with drift, Girsanov's theorem.

**2.**

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Let $W$ be a Brownian motion of a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t < \infty}, \mathbb{P})$. Show that, for each $x$, the process $M^x$ defined by

$$M_t^x = f(x + W_t), \quad 0 \leq t < \infty$$

is a local sub-martingale if and only if $f$ is convex.
   **Solution.** $\Leftarrow$ Suppose $f$ is convex. Let $0 \leq s < t < \infty$. Then

$$\mathbb{E}[M_t^x|\mathcal{F}_s] = \mathbb{E}[f(x + W_t)|\mathcal{F}_s] \geq f(\mathbb{E}[x + W_t|\mathcal{F}_s]) = f(x + W_s) = M_s^x.$$

So, $M_t^x$ is a local submartingale.
   $\Rightarrow$ Suppose $\{M_t^x\}$ is a local submartingale. Let $A, B \in \mathbb{R}$ and $\lambda > 0$. Choose $x, 0 \leq a < b$ such that such that $A = x + a$ and $B = x - b$ where and $\lambda = \frac{b}{a+b}$ and consider the stopping time $\tau = T_a \wedge T_{-b}$ where $T_a = \inf\{t \geq 0 : W_t = a\}$ and $T_{-b} = \inf\{t \geq 0 : W_t = -b\}$. Then, by the optional sampling theorem,

$$\mathbb{E}[M_\tau^x] \geq \mathbb{E}[M_0^x]$$

where

$$\mathbb{E}[M_\tau^x] = \frac{b}{a+b}f(x+a) + \frac{b}{a+b}f(x-b)$$

and

$$M_0^x = f(x) = f(\frac{b}{a+b}(x+a) + \frac{a}{a+b}(x-b)) = f(\frac{bx+ab+ax-ab}{a+b}) = f(x).$$

So, $f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda)f(B)$, so $f$ is convex.

**3.**

Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ where the filtration satisfies the usual conditions. On this space, consider two standard, one dimensional Brownian motions $W$ and $B$ (BM wrt the same filtration). Assume that $\langle B, W \rangle_t = \rho t$, where $\langle B, W \rangle$ is the cross variation of $B$ and $W$, and $\rho$ is constant. Show that

1. $B = W$, if $\rho = 1$

2. $B$ is independent of $W$ if $\rho = 0$.

   **Solution.**

1. Suppose $\langle B, W \rangle_t = t$. Consider the process $M_t = B_t - W_t$. Since $B_t, W_t$ are martingales, $M_t$ is a martingale. Then since $(B_t - W_t)^2 = B_t^2 - 2B_t W_t + W_t^2$ and we know $B_t^2 - t, W_t^2 - t$ and $B_t W_t - t$ are local martingales, $\langle B - W \rangle_t = 0$. Since $\langle B - W \rangle_t = \lim_{n\to\infty} \sum_{i=1}^{\infty} (B_{t_i^n \wedge t} - W_{t_i^n \wedge t} - B_{t_{i-1}^n \wedge t} + W_{t_{i-1}^n \wedge t})^2 = 0$, this implies $B - W$ has finite variation. Thus, since $B - W$ is a local martingale the finite variation, $B - W = 0$, so $B = W$.

2. Suppose $\langle B, W \rangle_t = 0$. By Lévy's characterization of Brownian motion, $(B, W)$ is a 2-dimensional Brownian motion $\Leftrightarrow B_t^2 - t, W_t^2 - t, B_t W_t$ are local martingales. Since $B, W$ are BM, $B_t^2 - t$ and $W_t^2 - t$ are local martingales. Since $\langle B, W \rangle_t = 0 = \frac{1}{2}(\langle B + w \rangle_t - \langle B \rangle_t - \langle W \rangle_t)$, $\langle B + W \rangle_t = \langle W \rangle_t + \langle B \rangle_t$. So, $B_t^2 + 2B_t W_t + W_t^2 - \langle B \rangle_t - \langle W \rangle_t$ is a local martingale, so $\{BW\}_t$ is a local martingale. Thus, $(B, W)$ is a 2-dimensional BM, so $B$ and $W$ are independent by the definition of 2-dimensional brownian motion.

# January 2019

**1.**

Let $W$ be a one-dimensional BM. Let $\mu, \sigma \in \mathbb{R}$ and $x$ an initial value. Solve in closed form the equation

$$\begin{cases} dX_t & = \mu X_t dt + \sigma X_t dW_t \\ X_0 = x. \end{cases}$$

**Solution.**

$$dX_t = X_t(\mu dt + \sigma dW_t)$$

From this form, we make a guess that $X_t = x \exp(\mu t + \sigma W_t)$ and check using Itô's formula.

$$dX_t = X_t d(\mu t + \sigma W_t) + \frac{1}{2} X_t d\langle \mu t + \sigma W_t \rangle_t = X_t(\mu dt + \sigma dW_t + \frac{1}{2}\sigma^2 dt) = X_t((\mu + \frac{1}{2}\sigma^2)dt + \sigma dW_t)$$

So, we need a correction of $-\frac{1}{2}\sigma^2 t$. Thus, $X_t = x \exp((\mu - \frac{1}{2}\sigma^2)t + W_t)$.

**2.**

Let $W$ be a standard one-dimensional Brownian motion and $M$ be its running maximum process, i.e.

$$M_t = \max_{0 \le s \le t} W_s, \quad 0 \le t < \infty.$$

Consider a two-times continuously differentiable function $f : \{(x, m) : m \ge 0, -\infty < x \le m\} \to \mathbb{R}$. Find a necessary and sufficient condition so that the process $Y$ defined by $Y_t = f(W_t, M_t), \quad 0 \le t < \infty$, is a local martingale.

   **Partial Solution.** For almost all $\omega$, the measure $dM(\omega)$ is singular with respect to the Lebesgue measure with support $\{t : W_t = M_t\}$. By Itô's formula,

$$dY_t = f_x(W_t, M_t)dW_t + f_m(W_t, M_t)dM_t + \frac{1}{2}f_{xx}(W_t, M_t)dt + f_{xm}(W_t, M_t)dW_t dM_t + \frac{1}{2}f_{mm}(W_t, M_t)dM_t dM_t$$

$$= f_x(W_t, M_t)dW_t + f_m(W_t, M_t)dM_t + \frac{1}{2}f_{xx}(W_t, M_t)dt$$

For $Y_t$ to be a local martingale, we need the finite variation parts to vanish, $f_{xx}(W_t, M_t) = 0$ everywhere and $f_m(W_t, M_t)$ on the support of $dM$, $f_m(W_t, M_t)$ on the diagonal $\{t : W_t = M_t\}$. These conditions are also sufficient.

**3.**

Consider a finite time horizon $T$ and a RCLL sub-martingale $(M_t)_{0 \le t \le T}$ on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $(\mathcal{F}_t)_{0 \le t \le T}$. Consider the optimization problem of finding the stopping time $\tau$ that maximizes the expected value $M$ at the (random) time $\tau$, namely the problem

$$\sup_{\tau \text{ a stopping time}} \mathbb{E}[M_\tau].$$

Find optimizer $\tau*$.

   **Solution.** Since $M_t$ is a RCLL sub-martingale, it admits a Doob-Meyer decomposition, $M_t = L_t + A_t$ where $L_t$ is a martingale and $A_t$ is a nondecreasing, predictable process. Since all viable stopping times are bounded by finite time horizon $T$, we can apply the bounded optional sampling theorem. For any stopping time $\tau$, $\mathbb{E}[M_\tau] = \mathbb{E}[L_\tau] + \mathbb{E}[A_\tau] = \mathbb{E}[L_0] + \mathbb{E}[A_\tau]$. Since $A_t$ is a nondecreasing process, this implies $\mathbb{E}[M_\tau]$ is maximized for $\tau = T$.

# August 2009

## 4.

Let $(B_t)_{0 \leq t < \infty}$ be a standard Brownian motion. Define the random time $T_x$ by

$$T_x = \inf\{t \geq 0 : B_t = x\}, \ x \in \mathbb{R}.$$

Compute $\mathbb{P}[T_a < T_{-b}]$ and $\mathbb{E}[T_a \wedge T_{-b}]$ for $a, b > 0$.

    **Solution.** Let $\tau = T_a \wedge T_{-b}$. Then, since $(B_t^\tau)_{0 \leq t < \infty}$ is bounded, $(B_t^\tau)$ is a uniformly integrable martingale. Thus, by the optional sampling theorem,

$$\mathbb{E}[B_\tau] = a\mathbb{P}[T_a < T_{-b}] + -b(1 - \mathbb{P}[T_a < T_{-b}]) = (a+b)\mathbb{P}[T_a < T_{-b}] - b = \mathbb{E}[B_0] = 0$$

which implies

$$\mathbb{P}[T_a < T_{-b}] = \frac{b}{a+b}.$$

    Since $(B_t)$ is a Brownian motion, $B_t^2 - t$ is a martingale. Again, applying the bounded optional sampling theorem,

$$\mathbb{E}[B_{t \wedge \tau}^2] = \mathbb{E}[t \wedge \tau].$$

Then letting $t \to \infty$, by the bounded convergence theorem for the LHS and the monotone convergence theorem for the RHS,

$$\mathbb{E}[B_\tau^2] = \mathbb{E}[\tau]$$

$$\mathbb{E}[\tau] = \mathbb{E}[B_\tau^2] = a^2 \mathbb{P}[T_a < T_{-b}] + b^2(1 - \mathbb{P}[T_a < T_{-b}])$$
$$= (a^2 - b^2)\left(\frac{b}{a+b}\right) + b^2 = (a-b)(a+b)\frac{b}{a+b} + b^2 = (a-b)b + b^2 = ab.$$

## 5.

Let $(B_t)$ be a standard Brownian motion. Show that

$$\lim_{t \to \infty} \sqrt{t}\mathbb{P}[B_s \leq 1, \forall \ s \leq t] = \sqrt{\frac{2}{\pi}}.$$

    **Solution.** Let $M_t$ be the running maximum process of the Brownian motion. Then

$$\mathbb{P}[B_s \leq 1, \ \forall \ s \leq t] = \mathbb{P}[M_t \leq 1] = \mathbb{P}[|B_t| \leq 1]$$

since $M_t \sim |B_t|$. So,

$$\lim_{t \to \infty} \sqrt{t}\mathbb{P}[B_s \leq 1, \forall \ s \leq t] = \lim_{t \to \infty} \sqrt{2}\mathbb{P}[|B_t| \leq 1] = \lim_{t \to \infty} \sqrt{t}\frac{2}{\sqrt{2\pi t}}\int_0^1 \exp(-x^2/2t) \ dx$$

$$= \frac{2}{\sqrt{2\pi}} \lim_{t \to \infty} \int_0^1 \exp(-x^2/2t) \ dx = \sqrt{\frac{2}{\pi}} \int_0^1 \lim_{t \to \infty} \exp(-x^2/2t) \ dx = \sqrt{\frac{2}{\pi}}.$$

where we can interchange the limit and integral due to the bounded convergence theorem.

## 6.

Let $(B_t)_{t \in [0,\infty)}$ be a B, and let $(X_t)_{t \geq 0}$ be its Lévy transform given

$$X_t = \int_0^t sgn(B_u)dB_u,$$

1. Show that $X$ is a Brownian motion.

2. Show that the random variables $B_t$ and $X_t$ are uncorrelated.

3. Show that $B_t$ and $X_t$ are not independent.

    **Solution.**

1. To show that $X$ is a Brownian motion, we will show that $X_t$ is a martingale and $\langle X \rangle_t = t$. Let $0 \leq s < t < \infty$. Consider a sequence of partitions of $[0,\infty)$ denoted $\Delta_n \to$Id. Then for $X_t^n = \sum_{k=1}^n sgn(B_{t_k} \wedge t)(B_{t_k \wedge t} - B_{t_{k-1} \wedge t})$, we consider the conditional expectation,

$$\mathbb{E}[X_t^n | \mathcal{F}_s] = \sum_{k=1}^\infty sgn(B_{t_k \wedge t})\mathbb{E}[(B_{t_k \wedge t} - B_{t_{k-1} \wedge t})|\mathcal{F}_s] = \sum_{k=1}^\infty sgn(B_{t_k \wedge s}))(B_{t_k \wedge s} - B_{t_{k-1} \wedge s}) = X_s^n.$$

    So $X^n$ is a martingale. Then $X_t^n \to X_t \ \forall \ t \in [0,\infty)$, so $X$ is also a martingale.

    Or, if we have that $X_t$ is a semi-martingale, since $X_t = \int_0^t sgn(B_s)dB_s$, $dX_t = sgn(B_t)dB_t$, so $X_t$ is a local martingale.

    Now, we want $\langle X \rangle_t$.

$$d\langle X \rangle_t = dX_t dX_t = (sgn(B_t)dB_t)(sgn(B_t)dB_t) = dB_t dB_t = d\langle B \rangle_t = dt$$

    So, $\langle X \rangle_t = t$. Thus, by Lévy's characterization of BM, $X$ is a Brownian Motion.

2. To show that $B_t$ and $X_t$ are uncorrelated, $\text{Cov}(X_t, B_t) = 0$ First, consider

$$X_t B_t = \int_0^t X_u dB_u + \int_0^t B_u dX_u + \int_0^t sgn(B_u)du.$$

The first two terms are local martingales with quadratic variations in $\mathbb{L}^2$,

$$\langle \int_0^t X_u dB_u \rangle = \int_0^t X_u^2 du$$

$$\langle \int_0^t B_u dX_u \rangle = \int_0^t B_u^2 du$$

so they are true martingales. So,

$$\text{Cov}(X_t B_t) = \mathbb{E}[X_t B_t] = \mathbb{E}[\int_0^t X_u dB_u + \int_0^t B_u dX_u + \int_0^t sgn(B_u)du] = \mathbb{E}[\int_0^t sgn(B_u)du] = \int_0^t \mathbb{E}[sgn(B_u)]du = 0.$$

3. To show that $B_t$ and $X_t$ are not independent, we calculate $\mathbb{E}[X_t B_t^2]$. First, since $dB_t^2 = 2B_t dB_t + dt$ by Itô,

$$X_t B_t^2 = 2\int_0^t X_u B_u dB_u + \int_0^t X_u du + \int_0^t B_u^2 dX_u + 2\int_0^t B_u sgn(B_u)du.$$

$\int_0^t X_u B_u dB_u$ and $\int_0^t B_u^2 dX_u$ are true martingales, so

$$\mathbb{E}[X_t B_t^2] = \mathbb{E}[\int_0^t (X_u + 2B_u sgn(B_u))du] = \mathbb{E}[\int_0^t (X_u + 2|B_u|)du] = \int_0^t \mathbb{E}[X_u + 2|B_u|]du.$$

Since $\mathbb{E}[X_u] = 0$,

$$\mathbb{E}[X_t B_t^2] = 2\int_0^t \mathbb{E}[|B_u|]du > 0.$$

However, $\mathbb{E}[X_t]\mathbb{E}[B_t^2] = 0$, so $\text{Cov}(X_t, B_t^2) \neq 0$ which implies $X_t$ and $B_t$ are not independent.

# January 2010

## 5.

Let $\{B_t\}_{t\in[0,\infty)}$ be a standard Brownian Motion and let $\{H_t\}_{t\in[0,\infty)}$ be a progressively measurable process such that

$$\forall\, t \geq 0, \quad \int_0^t H_u^2 du < \infty, \text{ and } \int_0^\infty H_u^2 du = \infty.$$

For $\sigma > 0$, show that $\int_0^\tau H_u dB_u \sim \mathcal{N}(0, \sigma^2)$, where $\tau = \inf\{t \geq 0 : \int_0^t H_u^2 du = \sigma^2\}$.

**Solution.** Let $M_t = \int_0^t H_u dB_u$. Since $H_t$ is progressively measurable with the above properties, $M_t$ is a local martingale. Then $\langle M \rangle_t = \int_0^t H_u^2 du$. Consider the exponential martingale

$$X_t = \exp(iuM_t + \frac{1}{2}u^2\langle M \rangle_t).$$

Then since $M_t^\tau$ is bounded and therefore uniformly integrable, we may apply the optional sampling theorem

$$\mathbb{E}[\exp(iuM_\tau)] = \frac{1}{\exp(\frac{1}{2}u^2\sigma^2)} = \exp(-\frac{1}{2}u^2\sigma^2)$$

Since the characteristic function of a normal random variable is $\varphi(t) = \exp(i\mu t - \frac{1}{2}\sigma^2 t^2)$, this implies $\int_0^\tau H_u\, dB_u \sim \mathcal{N}(0, \sigma^2)$.

# August 2010

## 3.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\mathcal{C}$ be a non-empty family of sub-$\sigma$-algebras of $\mathcal{F}$. For a random variable $X$ in $\mathcal{L}^1$, prove that the family

$$\chi = \{\mathbb{E}[X|\mathcal{G}] : \mathcal{G} \in \mathcal{C}\}$$

is uniformly integrable.

**Solution.** The family $\{X\}$ is uniformly integrable, so there exists a convex, nondecreasing test function $\varphi$ such that $\mathbb{E}[\varphi(X)] < \infty$. Then for $\mathcal{G} \in \mathcal{C}$. $\mathbb{E}[\varphi(\mathbb{E}[X|\mathcal{G}])] \leq \mathbb{E}[\mathbb{E}[\varphi(X)|\mathcal{G}]] = \mathbb{E}[\varphi(X)] < \infty$. Thus, $\chi$ is uniformly integrable.